Fuzzy Spatial Reasoning

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Abstract

We present a fuzzy version of the crisp spatial logic developed by Randell et al., which takes the single relation connected–with *as primitive. Membership functions are defined for each spatial relation defined in the crisp theory. Furthermore, principles are presented for defining linguistic variables whose linguistic values are spatial relations. The work reported here addresses spatial reasoning in situations where numerical or geometric precision is unlikely; it is particularly suited for dynamic situations.*

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1. Introduction

Spatial Reasoning is central to several important areas of engineering and computer science, including computer vision, assembly, and especially path planning for mobile robots. Because of the sort of mathematics forming the ubiquitous background of engineers, it is natural that engineers should attempt to represent spatial relations in two– or three–dimensional Euclidean space with values from the real number system or vector spaces over the reals or, in certain domains, with geometric structures. The data available in many situations, however, are not sufficiently complete or precise to support numerical or geometric descriptions that truly reflect the actual state of affairs. Furthermore, if human observers or controllers are involved, the data are likely in a linguistic form that is only indirectly related to numerical or geometric representations. The route of escape from this unnecessary and often spurious tyranny of precision without abandoning rigor is in the direction of other, often more modern, areas of mathematics, such as topology, abstract algebra (especially lattice theory and Boolean algebras), and formal logic. Although these areas avoid imposing notions of precision that are out of place, they reveal, often more perspicuously than does traditional engineering mathematics, relations of consistency, inconsistency, and entailment. Further insight into spatial relations has arisen from ontology, the study of the basic categories of what exists, which is traditionally an area of philosophy [7] and has recently been extensively applied in artificial intelligence [6]. Within ontology, we are particularly concerned with the areas called mereology, which addresses the part–of (or part–whole) relation, and "topology", which addresses the connected–with relation and, as the term is used here, uses different methods to address roughly the same area as that addressed by algebraic topology in mathematics [8]. Following Randell, Cohn, and Cui [2,7], we shall take the connected–with relation as primitive and derive from it an entire system of spatial relations, including the part–of relation. The entire system is expressed in first–order predicate logic and includes special axioms from which the fundamental properties of interest are derived. This crisp theory involves no numbers and does not directly support geometric properties. Yet it allows one to reason about spatial relations and what sequences of these relations are possible as a system evolves.

The crisp theory of spatial relations, however, is not entirely free from dependence on precision, for many of the relations occur at thresholds and all of them may shade imperceptibly into other relations. This dependence on precision becomes onerous when the theory is applied to complex systems, about which we have limited ability to make precise statements if only because we usually begin with imprecise data. This motivates introduction of a fuzzy version of the Randell *et al*.'s spatial logic since fuzzy set theory and logic adapt crisp notions from set theory and logic so that they may accommodate imprecision and approximation [9]. We do this, however, in a principled way, so as not to compromise rigor. The general principle followed here is to develop fuzzy logics from already proven crisp logics in a way that preserves their desirable properties.

Much of the appeal of a logical formalism is its close correspondence with natural language. This is preserved with the fuzzy version. Indeed, this correspondence is enhanced by the linguistic variables of the fuzzy version as long as the values are motivated by the logical structures themselves and are not derived solely from convenience in representing membership functions.

The remainder of this paper is organized as follows. In section 2, we present the basics definitions of Randell *et al.*'s crisp theory and the definitions of the membership functions for the fuzzy version. Section 3 shows how linguistic variables should be defined for the fuzzy theory. Section 4 briefly addresses the methods of reasoning that are hereby enabled, and section 5 concludes and addresses future work.

2. The Basic Definitions

We assume one dyadic (two–argument) relation, $C(x,y)$, understood to mean that x is connected with y. This relation is taken to include any kind of spatial contiguity, include the cases where x is a part of y, where x and y overlap, and where the boundaries of x and y share a common point but x and y have disjoint interiors. In terms of point-set topology, $C(x,y)$ holds when the closures of regions x and y share a common point. The C relation and the other relations to follow are sufficiently abstract that the individuals x and y can be thought of as several kinds of things, for example, physical objects, regions in space, and even temporal regions. To form examples of theses relations, it is helpful to think of the individuals as either spatial regions with physical boundaries (such as rooms) or objects that can be located in such regions (such as desks in rooms).

Two properties of the connected–with relation are immediately evident. First of all, it is reflexive: anything is connected with itself, or, in symbols, $\forall x\ C(x,x)$. Secondly, the connected–with relation is symmetric: if x is connected with y, then y is connected with x, or, in symbols, $\forall xy\ [C(x,y) \rightarrow C(y,x)]$.

The translation of these properties to membership functions for dyadic fuzzy relations is straightforward. We assume a universe U of individuals to which our spatial relations apply. Then, where $U \times U$ is the Cartesian product of U with itself (that is, the set of all ordered pairs where both the first and the second elements are in U), any dyadic fuzzy relation R on U has the form

$$R = \{((x,y), \mu_R(x,y)) \mid (x,y) \in U \times U\}$$

That is, R is viewed as the set of pairs (x,y), with $x \in U$ and $y \in U$, each pair itself paired with a value of the membership function μ_R for R. We assume that every fuzzy relation R we work with is normal, that is, $0 \le \mu_R(x,y) \le 1$ for all $x,y \in U$. We require that the crisp version of the spatial logic be recoverable from the fuzzy version. To that end, we define the crisp relation corresponding to the fuzzy relation R as the strong 0.5–cut $R_{0.5}$ of R:

$$R_{0.5} = \{(x,y) \in U \times U \mid \mu_R(x,y) > 0.5\}$$

That is, the relation $R_{0.5}$ holds from x to y (or between x and y, in that order) if and only if $\mu_R(x,y) > 0.5$. Note that $R_{0.5}$ is then a subset of $U \times U$, as required of a crisp relation over U. Translating to the crisp version, then, amounts to taking a formula of the form $R'(a,b)$ (a is related by R' to b – for example, a is connected with b) and considering it true if $\mu_R(a,b) > 0.5$, where R is the fuzzy relation corresponding to R'. If $\mu_R(a,b) \le 0.5$, then $R'(a,b)$ is considered false. Thus, if we begin with a crisp version in which all statements of the form $R'(x,y)$ are true, we translate to the fuzzy version by considering a relation

$$R = \{((x,y), \mu_R(x,y)) \mid (x,y) \in U \times U\}$$

where we require $\mu_R(x,y) > 0.5$ if $R'(x,y)$ is true (or $(x,y) \in R'$). Thus, the natural way to express the reflexive property of the fuzzy connected–with relation is

$$\mu_C(x,x) > 0.5$$

where, as throughout, we use the same name for the fuzzy relation as for the crisp relation – context clearly disambiguates.

Next, the natural way to view reflexivity is as a guarantee that the order of the arguments makes no difference. Thus, for the fuzzy connected–with relation, we require that

$$\mu_C(x,y) = \mu_C(y,x)$$

Given the connected–with relation $C(x,y)$, we can define a basic set of dyadic spatial relations. First of all, we say that x is disconnected from y (written $DC(x,y)$) if x is not connected with y:

$$DC(x,y) =_{def} \neg C(x,y)$$

Since we require our fuzzy relations to be normal, the appropriate notion of negation is complement with respect to 1. Thus we require of the fuzzy DC relation that

$$\mu_{DC}(x,y) = 1 - \mu_C(x,y)$$

Next, x is a part of y, $P(x,y)$, if everything connected with x is also connected with y:

$$P(x,y) =_{def} \forall z\ [C(z,x) \rightarrow C(z,y)]$$

The corresponding constraint on the fuzzy relation P can be developed in a small number of steps. First, we think of a conditional $\varphi \rightarrow \psi$ in the manner of material implication as $\neg\varphi \vee \psi$. As is usual, we relate the *OR*, $\vee$, with the maximum of the membership functions. Similarly, we relate the *AND*, *Ù*, with the minimum of the membership functions. A universal quantification *"x j(x)* is thought of as the conjunction of all substitution instances of *j(x)*: *j(u₁) Ù j(u₂) Ù ¼ Ù j(uₙ)*, where $U = \{u_1, u_2, ¼, u_n\}$. Thus, universal quantification is related with the minimum value of the argument as the variable bound by the quantifier ranges over the elements of the universe. Similarly, an existential quantification *$x j(x)* is thought of as the disjunction of all substitution instances of *j(x)*: *j(u₁) Ú j(u₂) Ú ¼ Ú j(uₙ)*. Thus, existential quantification is related with the maximum value of the argument as the bound variable ranges over the elements of the universe. Following these principles, we require for the fuzzy relation *P* that

$$m_P(x, y) = \min_{z \in U}\left\{\max\left\{1 - m_C(z, x), m_C(z, y)\right\}\right\}$$

The definitions of the remaining crisp relations can be related to constraints on the membership functions of the corresponding fuzzy relations following the principles articulated above. We now list these relations without further comment. For each relation, we first give the English reading, then the definition of the crisp relation, and finally the constraint on the corresponding fuzzy relation. Figure 1 portrays the more specific of the spatial relations.

x is a proper part of *y*
PP(x,y) $=_{def}$ *P(x,y) Ù ØP(y,x)*

$$m_{PP}(x, y) = \min\left\{m_P(x, y), 1 - m_P(y, x)\right\}$$

x is identical with *y*
x = y $=_{def}$ *P(x,y) Ù P(y,x)*

$$m_{=}(x, y) = \min\left\{m_P(x, y), m_P(y, x)\right\}$$

x overlaps *y*
O(x,y) $=_{def}$ *$z [P(z,x) Ù P(z,y)]*

$$m_O(x, y) = \max_{z \in U}\left\{\min\left\{m_P(z, x), m_P(z, y)\right\}\right\}$$

x partially overlaps *y*
PO(x,y) $=_{def}$ *O(x,y) Ù ØP(x,y) Ù ØP(y,x)*

$$m_{PO}(x, y) = \min\left\{m_O(x, y), 1 - m_P(x, y), 1 - m_P(y, x)\right\}$$

x is discrete from *y*
DR(x,y) $=_{def}$ *ØO(x,y)*

$$m_{DR}(x, y) = 1 - m_O(x, y)$$

x is externally connected with *y*
EC(x,y) $=_{def}$ *C(x,y) Ù ØO(x,y)*

$$m_{EC}(x, y) = \min\left\{m_C(x, y), 1 - m_O(x, y)\right\}$$

x is a tangential proper part of y
TPP(x,y) $=_{def}$ *PP(x,y) Ù $z [EC(z,x) Ù EC(z,y)]*

$$m_{TPP}(x, y) = \min\left\{m_{PP}(x, y), \max_{z \in U}\left\{\min\left\{m_{EC}(z, x), m_{EC}(z, y)\right\}\right\}\right\}$$

x is a nontangential proper part of *y*
NTPP(x,y) $=_{def}$ *PP(x,y) Ù Ø$z [EC(z,x) Ù EC(z,y)]*

$$m_{NTPP}(x, y) = \min\left\{m_{PP}(x, y), 1 - \max_{z \in U}\left\{\min\left\{m_{EC}(z, x), m_{EC}(z, y)\right\}\right\}\right\}$$

The relations *P*, *PP*, *TPP*, and *NTPP* are non-symmetric. For example, if *x* is a part of *y* (that is, *P(x,y)*), it does not follow that *y* is a part of *x*. Thus, the inverse of any of these relations is distinct from the relation itself. For example, asserting that *x* is a part of *y* asserts something different from asserting that *y* is a part of *x*. Where

F is any relation, we represent its inverse by F^{-1}. Concerning the constraints on the corresponding fuzzy relations, note that, in general,

$$m_{\Phi^{-1}}(x,y) = m_{\Phi}(y,x)$$

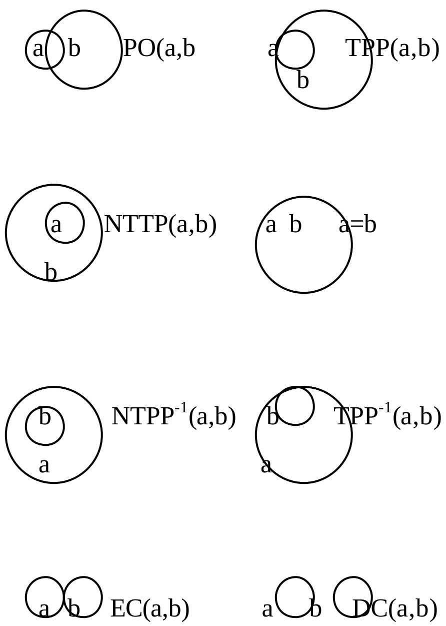

Figure 1. Some spatial relations

3. Linguistic Variables

As pointed out by Randell *et al*. [7], the entailment relations among the spatial relations induce a lattice structure on the family of spatial relations. We exploit this lattice to define spatial linguistic variables. Figure 2 shows the lattice of spatial relations. Where R_1 and R_2 are spatial relations, we write R_1 £$_L$ R_2 if R_1 is below R_2 in the lattice, that is, if there is a path from R_1 to R_2 going in the direction of the top, ⊤. Note that, for any relation R, we have R £$_L$ R, that is, $\leq_L$ is reflexive. For any relations R_1 and R_2 such that R_1 £$_L$ R_2, we have

"xy [$R_1(x,y)$ ® $R_2(x,y)$]

The top relation in the lattice, ⊤, is the tautologous relation, which holds of any pair of elements: *"xy ⊤(x,y)*. The bottom relation, ^, is the contradictory relation, which holds of no pair of elements: *"xy Ø^(x,y)*.

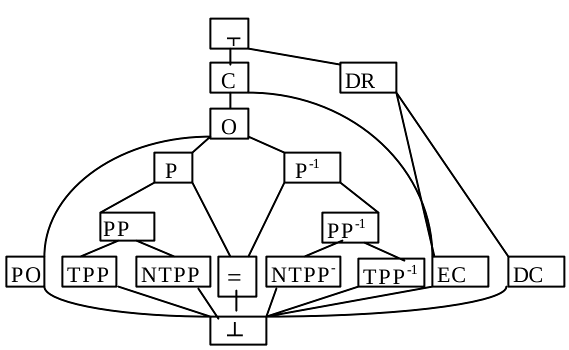

Figure 2. The lattice of spatial relations [7].

Now, let us define a *cutset* of the lattice to be a set of relations that has as a member one relation from every path between ^ and ⊤. Examples of cutsets are *{C, DR}*, *{PO, P, P^{-1}, EC, DC}*, and *{PO, TPP, NTPP, =, $NTPP^{-1}$, TPP^{-1}, EC, DC}*. The latter set is of particular note since it comprises the relations immediately above ^; we denote it by *B*. It can be shown that the members of any cutset are exhaustive and pairwise disjoint [7]. That is, given a pair *x, y* of individuals, exactly one relation in a cutset holds of that pair.

Turning to the corresponding fuzzy relations, the exhaustive and exclusive nature of the relations in a cutset is reflected in the fact that, given any pair *x, y* of individuals, the value $m_R(x,y)$ of the member function of one relation *R* for *x* and *y* tends to clearly dominate the values for *x* and *y* of the member functions of the other relations in the cutset as long as the crisp condition is approached. As imprecision and uncertainty take over, of course, this dominance disappears and there may be several contenders for the dominant relation. The characteristics of this dominance motivate defining a linguistic variable corresponding to $\top$ whose linguistic values are the relations in a given cutset. Such a linguistic variable acts as a classifier; the cutset selected for the linguistic values determines how finely and in what respects we classify. In particular, if *B* is used, we have the finest classifier supported by this taxonomy of spatial relations.

We can define linguistic variables that are restricted to a part of the lattice by introducing the notion of an *R-cutset*, where *R* is a spatial relation. An *R*-cutset is simply a cutset of the sublattice consisting of those relations R_1 such that R_1 £$_L$ R_2; note that *R* is the top of the sublattice. For example, there are two *P*-cutsets: *{PP, =}* and *{TPP, NTPP, =}*. Given a relation *R*, we can define a linguistic variable whose linguistic values are the relations in a given *R*-cutset. Such a linguistic variable acts as a restricted classifier.

Use of linguistic variables in a principled way, such as here, offers several advantages. For one thing, linguistic variables greatly facilitate human input and allow output that is easily interpreted by humans. Again, linguistic variables support discrete classification, allowing a system to focus on the most significant aspects. Using restricted linguistic variables, based on *R*-cutsets, further supports the ability of a system to focus on significant aspects. Finally, allowing different cutsets or R-cutsets to supply linguistic values supports flexibility while retaining access to all relations in the taxonomy.

4. Reasoning with the Fuzzy Spatial Logic

Randell *et al*. support reasoning with the crisp spatial logic by using a theorem prover for many-sorted logic [7]. Our approach with the fuzzy version, in contrast, exploits the connections among the membership functions. These connections form a network of constraints for which we are implementing constraint satisfaction methods.

Randell *et al*. have used their spatial logic for qualitative simulation of dynamic situations [6]. Figure 3 shows the allowable transitions among the relations of cutset *B*; only these transitions should occur as two bodies move or otherwise change in relation to each other. The same transitions are allowable for the fuzzy relations, but now precise thresholds need not be enforced. We are particularly interested in applications to dynamic path planning for mobile robots.

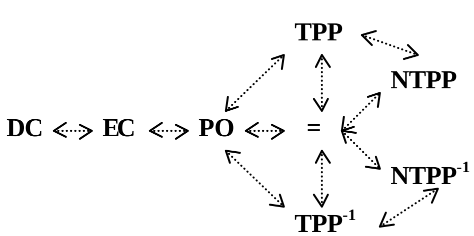

Figure 3. Transitions among relations

5. Conclusions and Future Work

We have presented a fuzzy version of the crisp spatial logic developed by Randell *et al*. that takes the single relation connected–with as primitive. Membership functions have been defined for each spatial relation defined in the crisp theory. Furthermore, principles have been presented for defining linguistic variables whose linguistic values are spatial relations.

To apply our results, we must develop principles for how various information sources – such as sensors and human speech – can contribute to the fuzzy measures. To exploit our results, we are developing constraint satisfaction methods to apply to the constraint network induced by the definitions of the membership functions. For such methods to be feasible, we need ways to limit the number of individuals considered. This is particularly critical for cases where the crisp definitions include quantifiers, for then we take maxima and minima over all the individuals in our universe of discourse. Intuitively, however, one should be able to exploit known spatial relations to safely ignore the vast majority of individuals in any one case.

The spatial logic of Randell *et al*. is only one of several AI logics for which useful fuzzy versions can be developed. We intend to address Allen's logic of temporal intervals [1] and Galton's logic of space, time, and motion [3].

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