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Relation Algebras over Containers and Surfaces: An Ontological Study of a Room Space*

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Abstract

Recent research in geographic information systems has been concerned with the construction of algebras to make inferences about spatial relations by embedding spatial relations within a space in which decisions about compositions are derived geometrically. We pursue an alternative approach by studying spatial relations and their inferences in a concrete spatial scenario, a room space that contains such manipulable objects as a box, a ball, a table, a sheet of paper, and a pen. We derive from the observed spatial properties an algebra related to the fundamental spatial concepts of containers and surfaces and show that this container-surface algebra holds all properties of Tarski's relation algebra, except for the associativity. The crispness of the compositions can be refined by considering the relative size of the objects) and their roles (i.e., whether it is explicitly known that the objects are containers or surfaces).

1. Introduction

We study the inference power of relation algebras related to the two basic spatial concepts of containers and surfaces. Algebras over spatial relations provide powerful mechanisms for spatial reasoning [6, 8, 40]. Such inferences have become increasingly important in geographic information systems where people record field observations [13, 33] and later analyze them to find new information [7]. Unlike calculations that employ traditional computational geometry, spatial-relation algebras rely on symbolic computations over small sets of relations [3]. This method is

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very versatile since no detailed information about the geometry of the objects, such as coordinates of boundary points or shape parameters, is necessary to make inferences. The domain for these algebras are qualitative spatial relations, which are predicates that come close to what people use in everyday reasoning about geographic space [5]. A large variety of formalisms exist for different types of spatial relations, such as topological relations [4, 37], cardinal directions [9, 10, 14, 36], approximate distances [20, 22], and their combinations [19, 35, 39]. While these formalisms capture semantics in a mathematically sound way, it has been questioned whether their inferences are cognitively plausible [11, 19]. Information systems that generate intuitive answers are better accepted and commonly desired [26, 32]. Studies of spatial commonsense concepts for defining spatial relations have been limited. Models of spatial relations described by linguistics and psychologists are based upon introspection [21, 41] and usually lack the formalism necessary to be used in information systems.

In this paper, we resort to the relations associated with two fundamental spatial concepts—a container and a surface. A container affords putting things into and out of it, whereas a surface enables support (i.e., to put things onto and off the surface). The spatial concepts of containers and surfaces are among the image schemata that were identified by cognitive linguists as basic categories of human experience [23, 28] and their importance was emphasized by a study of image schemata in children's books, which revealed that containers and surfaces are the first and most frequent spatial concepts taught [12]. A basis for a formalization of the semantics of these image schemata is an understanding of the behaviors of prototypical containers and surfaces. We are primarily interested in the semantics of the inferences that may be made with containers, surfaces, and combinations of containers and surfaces. For example, if an object is inside of a container that is on a surface, what are the possible relations between the object and the surface?

To avoid the assignment of arbitrary semantics, we start with a study of a concrete scenario that exposes a rich environment for spatial reasoning. Our choice is a room space, a small-scale space that contains manipulable objects and retains the bodily experiences of how people assimilate image schemata. For example, people experience the container schema by moving in or out of rooms and the surface schema by laying on their beds. They manipulate objects in a room space by placing objects into containers, such as boxes, cups, and drawers, and onto surfaces, such as tables, desks, and shelves. The study of the operations on objects in a room leads to an ontology of the room space. We borrow the term ontology from artificial intelligence, where an ontology is a repository or resource containing knowledge about concepts that exist in their world and the describable relations among them [15, 16]. We constrain these definitions of ontology to denote the identification of objects, their interrelations, and the way their interrelations may change within our specific case study. By doing so, this study follows Hayes's [18] approach to the definition of an ontology of liquids, analyzing the states and changes of states of particular real-world scenarios. Our study, however, concentrates on the description of solid objects rather than liquid substances and it is concerned with configurations in a room space that presents only the container and surface schemata. Other image schemata, in particular partial containment or support that result from a part-whole image schema, are beyond the scope of this work.

A formalization of the room-space ontology is done in terms of a relation algebra [42, 43]. A relation algebra defines relations as variables and operations as mechanism of inferences. A relation algebra satisfies properties over the combination of compositions, which are of great interest to assess when these combinations produce stable results. It is more flexible than other mechanisms, such as algebraic specifications which are strongly typed such that all objects participating in a spatial configuration must have a predetermined type. A relation algebra, on the other hand, provides mechanisms of inference that do not force all objects to be typed. For example, if an object A is *inside* of a container B that is *inside* of another container C, A is *inside* of C, regardless of the type of object A.

The remainder of this paper is organized as follows. Section 2 summarizes the main characteristics of a relation algebra. Section 3 presents a natural-language description of the ontology of a room space, which is the basis for the definition of the container-surface algebra in

Section 4. Section 5 analyzes the reasoning power of refinements of the container-surface algebra, considering the relative sizes and the roles of objects. Conclusions and future work are discussed in Section 6.

2. Relation Algebras

A *relation algebra* (with universe) is defined as a ten-tuple ,+, ·, −, 0, 1, ;,1', 0', ¾ where ,+, ·, −, 0, 1 is a Boolean algebra, 0 is the *empty* relation, 1 is the *universal* relation, ; is a binary operation called *composition*, 1' is the *identity* relation, 0' is the *diversity* relation, and ¾ is a unary operator forming the *converse* of a given relation [42]. Since a relation algebra is an extension of a Boolean algebra, its elements (i.e., relations) retain the associativity, distributivity, and DeMorgan's law. In addition, a relation algebra satisfies seven axioms among elements x, y, z (Equations 1a-g).

$$(x;y);z = x;(y;z) \tag{1a}$$
$$(x+y);z = x;z + y;z \tag{1b}$$
$$x;1' = x \tag{1c}$$
$$\overline{(\bar{x})} = x \tag{1d}$$
$$\overline{(x+y)} = \bar{x}+\bar{y} \tag{1e}$$
$$\overline{(x;y)} = \bar{y};\bar{x} \tag{1f}$$
$$\bar{x};-(x;y)+-y = -y \tag{1g}$$

The composition operation (;) allows us to make inferences about the relation between two objects, o_i and o_k, by combining the relations, R and S, with a third common object, o_j (Equation 2). The composition may result in a set of spatial relations that is composed of one or more than one element and whose number of elements increases as less precise information can be obtained from the inference. For example, if the composition of two operations results in a set with all possible relations (i.e., the universal relation), no information at all is obtained from this inference. A special case of composition is transitivity if R ; R implies R .

$$R\ ;S \equiv (o_i, o_k) \mid \quad o_j \quad \text{such that} \quad (o_i, o_j) \quad R \ \text{ and } (o_j, o_k) \quad S \tag{2}$$

A relation algebra is called *proper* if its universe is a set of binary relations and its operations coincide with the usual set-theoretic operations on these relations [27, 30]. This means that $x+y$ is defined as the union, $x \cdot y$ as the intersection, and $\bar{x}$ is the complement of x with respect to the universe. For expressions of relation algebras written without parenthesis, unary operators are executed first, followed by the composition, intersection, and union.

Varieties on the associative property of relation algebras have been studied [1, 31]. They define semiassociative (Equation 3a), weakly associative (Equation 3b), and nonassociative (Equation 3c) algebras. Relation algebras (RA), semiassociative algebras (SA), weakly associative algebras(WA), and nonassociative algebras (NA) are related by Equation 3d. Most theorems defined for a relation algebra still hold for a semiassociative algebra [2]. Furthermore, since many of these theorems do not require the semiassociative law, these theorems are also valid for weakly and nonassociative algebras.

$$x;(1;1) = (x;1);1 \tag{3a}$$
$$(1'\cdot x);1;1 = (1'\cdot x);1 \tag{3b}$$

$$x;y \cdot z = 0 \ iff \ \bar{x};z \cdot y = 0$$
$$y;x \cdot z = 0 \ iff \ z;\bar{x} \cdot y = 0 \qquad (3c)$$

$$RA \quad SA \quad WA \quad NA \qquad (3d)$$

3. Ontology of a Room Space

A room space is an ubiquitous example of a small-scale space, a space where people manipulate objects and experience objects from one standpoint [25, 34, 44]. As such, the room space constitutes a representative scenario where people experience recurrent manifestations of image schemata. This study considers a neatly organized room space with six major objects: a box, a ball, a table, a sheet of paper, a pen, and a room itself. All these items are solid objects that people manipulate by changing their locations. We assume, for the time being, that all objects may only be completely on or off a surface, i.e., no part of, say, the paper may extend beyond the tabletop. Similarly, all objects can be only completely in or out of a container. The discussion of partially on and off and partially inside and outside relates to the *part-whole* image schema, which would introduce more complex variations of the simpler but more fundamental cases considered here. Subsequently, the room is analyzed from simple to more complex configurations by describing the operations upon the objects and their interrelationships. In this ontology, rather than defining the objects themselves, the behavior of the objects is described in relation to changes of the spatial relations within the spatial configurations.

3.1 Spatial Configurations

Six spatial configurations were selected to provide a comprehensive treatment of situations between containers and surfaces, that is, they cover all possible combinations of objects with respect to containers and surfaces. These six selected configurations are:

- a ball in a box (object in a container),
- a box with a ball *inside* in a room (object in a container in another container),
- a paper on a table (object on a surface),
- a paper with a pen on top on a table (object on a surface on another surface),
- a box with a ball *inside* on a table (object in a container on a surface), and
- a table with a paper on the top in a room (object on a surface in a container).

Further configurations could be derived from combinations of these six scenarios. For example, a pen on a paper on a table in a room could be decomposed into an object on a surface (pen on paper) and an object on a surface in a container (paper on table in room).

For objects within these configurations, the analysis only considers operations that comprise movements that do not change upright or horizontal positions. Operations that change upright and horizontal positions may depend on particular characteristics of each object and may change the object's condition of being a projection of the container or surface schema.

3.2 Ball in a Box

The box serves as a container so that it can contain other objects. In our scenario the box is in an upright position as a prototypical situation and objects can be moved into it. The box is a container by creation, i.e., it does not need to have another object in it in order to be a container. There is some logic in saying that the box always contains something, be it pens, papers, or air. In this scenario, however, it is assumed that the box is empty if there is only air in it. In that case, the ball is outside the box. By moving the ball into the box, the ball is in the box (Figure 1a). The box can be moved and the ball moves with it (Figure 1b). The ball can be removed from the box and the ball would be outside of the box again (Figure 1c).

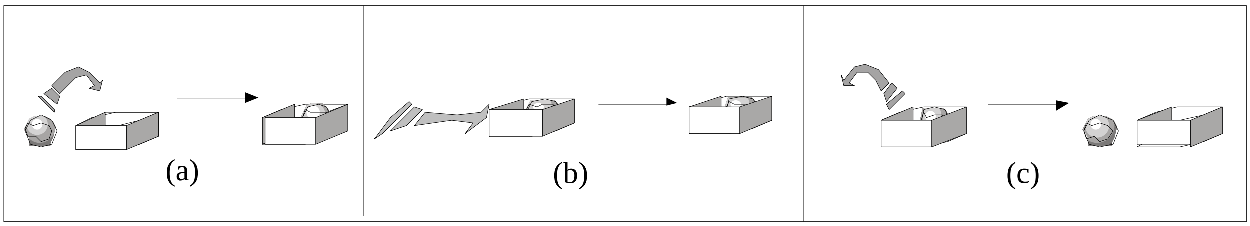


Figure 1: Ball and box: (a) moving ball into box; (b) moving box with ball around; (c) and removing ball from box.

3.3 Ball in a Box in a Room

The room acts as another larger container for the ball and the box, and the same operations and properties apply to box and room, and ball and room, as did apply to ball and box. The interesting aspect here is, however, the interplay between the ball (in the box) and the room. Moving the box (with the ball) into the room means that the ball is in the room as well (Figure 2a). Moving the box that contains a ball around the room keeps the same relations between ball and box, and ball and room (Figure 2b). If the box with the ball is moved out of the room, however, the ball goes with it (Figure 2c). Moving the ball into the box that is already in the room means to move the ball into the room and then into the box (Figure 2d). Reversibly, if the ball is removed from the box while the box is in the room, the ball would be still in the room, at least for some non-zero time (Figure 2e). Finally, removing the ball that is in the box from the room implies to remove the ball from the box and then from the room (Figure 2f).

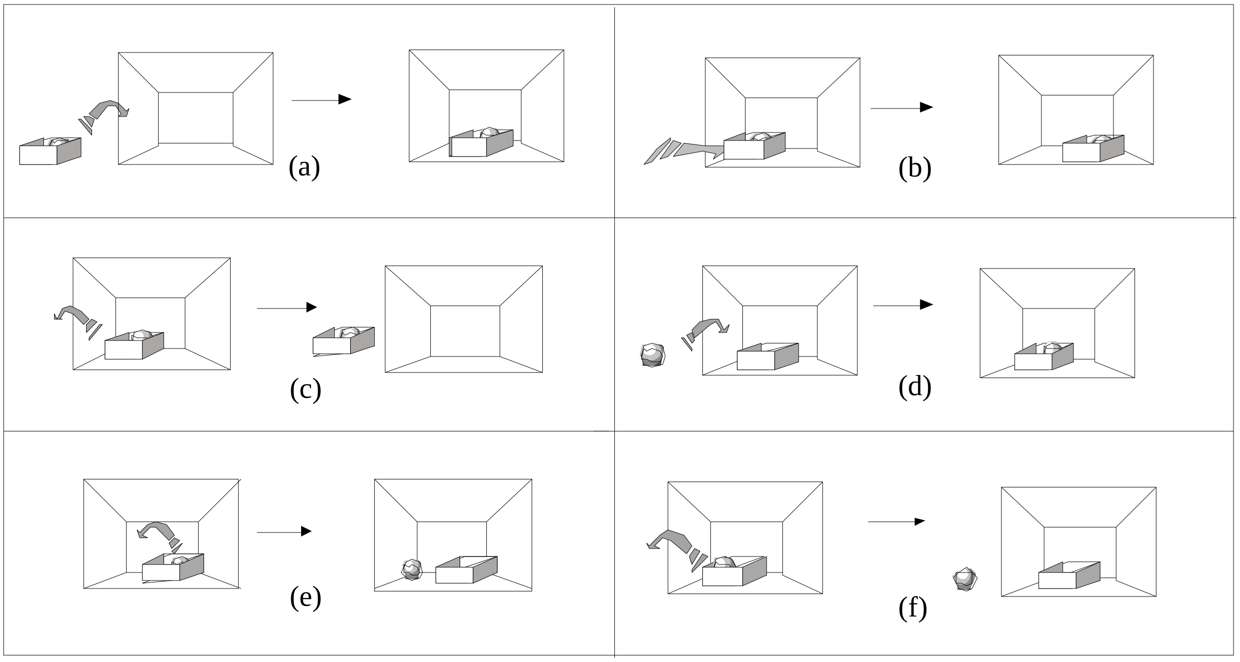


Figure 2: Ball, box, and room: (a) moving box with ball into room; (b) moving box with ballaround room; (c) removing box with ball from room; (d) moving ball into box in room; (e) removing ball from box in room; and (f) removing ball in box from room.

3.4 Paper on a Table

The table acts as a surface. Similar to the box, which is a container by creation, the table is a surface by creation and in its horizontal position, one can put objects onto it. If the sheet of paper is put on the table, the paper and the table have contact (Figure 3a). If the paper is moved around on the table, it stays in contact with the table (Figure 3b). If it is removed from the table, the paper loses its contact with the table and is off the table (Figure 3c). The table with the paper may be moved around such that the paper might remain on or move off the table. This last case is distinguished by saying that the table has been moved quickly—called *jerked*—such that the paper is also removed from the table (Figure 3d).

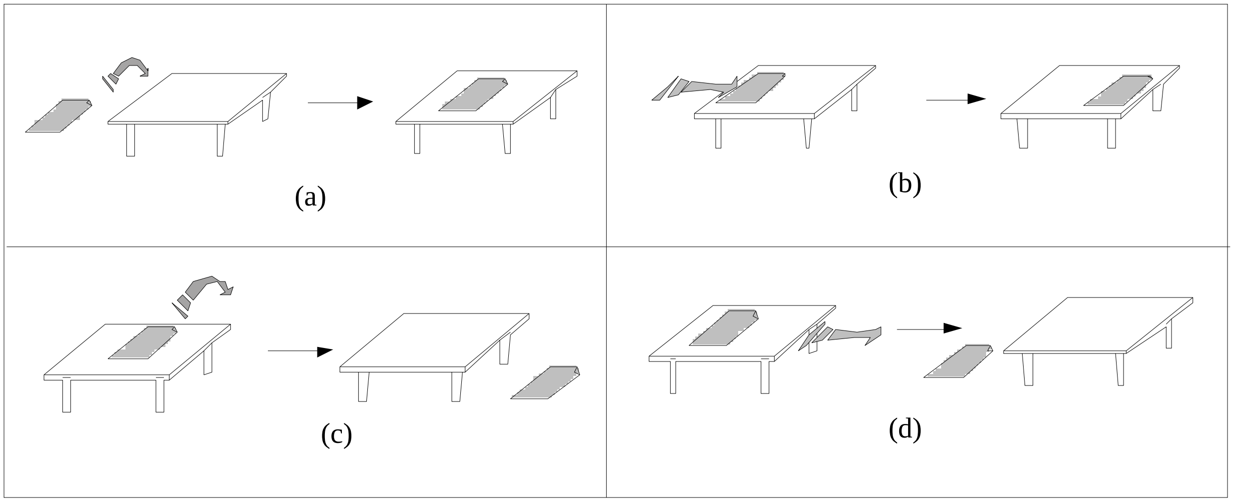


Figure 3: Paper and table: (a) putting paper onto table; (b) moving paper around table; (c) removing paper from table; and (d) jerking table with paper.

3.5 A Pen on a Paper on a Table

A paper with a pen may be put onto a table (Figure 4a). The same configuration results from moving the pen onto the paper that is already on the table (Figure 4b). Although the same operations and properties apply to pen and paper as did apply to the paper and table, the interesting aspect of this configuration is the relationship between the pen (on the paper) and the table. While the pen is now on the table, it does not have contact with the table. Being on the surface is transitive, having contact is not. If the paper is moved around on the table, the pen will either move with it or end up being in contact with the table. These two situations are distinguished by saying that the paper has been moved around (Figure 4c) or has been jerked (Figure 4d). Likewise, the table may be moved around such that the same properties between pen and paper, and pen and table are preserved (Figure 4e), or the table may be jerked so that the paper with the pen is removed from the table (Figure 4f). Although the pen that is on the paper might move off the paper when the table is jerked, it is assumed that the pen remains on the paper until an operation “remove pen from paper” is performed. The paper may be removed from the table such that it keeps the pen on it (Figure 4g). The pen on a paper alone may be moved such that it remains on the paper as long as it keeps his contact with the paper. A pen that is moved horizontally and loses its contact with the paper may remain on the table as long as it lays on the table. The pen, however, may have been removed off the table as it was moved off the paper, because the paper and the table were aligned along a common edge and the pen was moved across the edge (Figure 4h). For simplicity, it is assumed that moving the pen such that it loses contact with the paper is an operation that removes the pen from the paper without any assumption that the pen will end up on the table (Figure 4i). If the pen then lays on the table, it would mean there was an operation “put pen onto table” (Figure 4j).

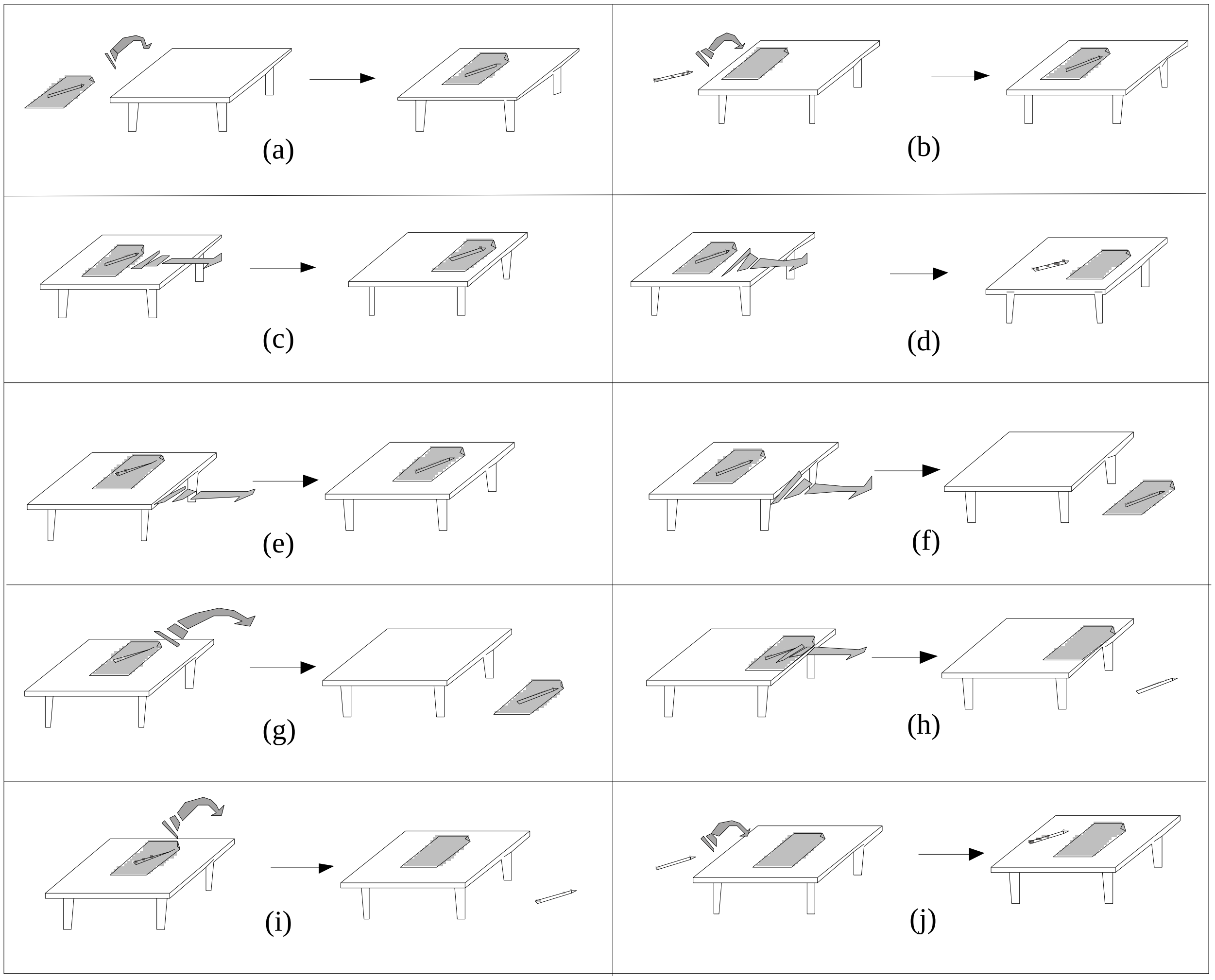


Figure 4: Pen, paper, and table: (a) putting paper with pen onto table; (b) putting pen onto paper on table; (c) moving paper with pen around table; (d) jerking paper with pen on table; (e) moving table with box and ball around; (f) jerking table with box and ball; (g) removing paper with pen from table; (h) moving pen cross edge of table; (i) removing pen from paper on table; and (j) putting pen onto table.

3.6 Ball in a Box on a Table

A box with a ball may be moved onto a table such that the ball is on the table, without having contact with the table (Figure 5a). While the box is moved around on the table, the ball remains in it (Figure 5b). If the box is removed from the table, the ball will move with it (Figure 5c). The ball may be removed from the box that is on the table so that the ball is no longer on the table (Figure 5d). The table may be moved and the box with the ball might either remain on or move off the table. Like in the previous cases (Sections 3.4 and 3.5), these two situations are distinguished by saying that the table has been moved around (Figure 4e) or jerked (Figure 4f). There are different judgments, however, about the relation between the ball and table. While some people might argue the ball is on the table, others might argue the ball is not on the table, because it does not have direct interaction with the table. This situation suggests a context dependence of spatial relations, which is related to Herskovits's [21] discussion about cognitive aspects of locative prepositions. Here we assume that the ball is on the table to explore a richer inference mechanism.

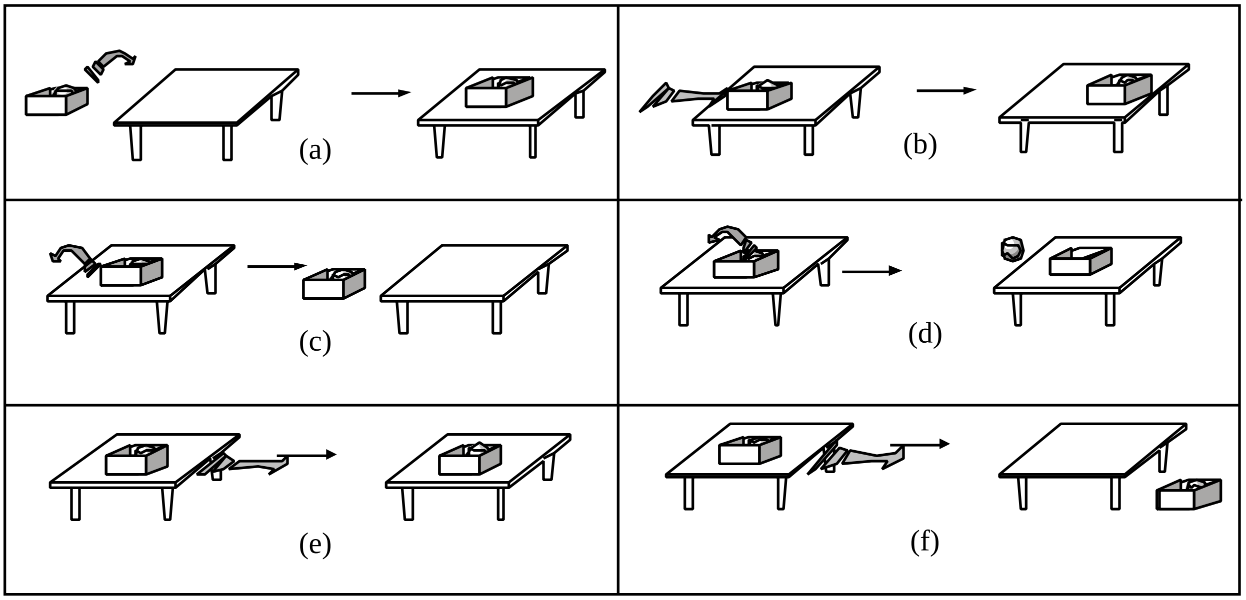


Figure 5: Ball, box, and table: (a) putting box with box onto table; (b) moving box with ball around table; (c) removing box with ball from table; (d) removing ball from box on table; (e) moving table with box and ball around; and (f) jerking table with box and ball.

3.7 Paper on a Table in a Room

A sheet of paper on a table may be moved into a room (Figure 6a). The sheet of paper may also be moved onto the table when the table is already in the room such that the paper is moved into the room and then onto the table (Figure 6b). Because the table is in the room, the paper on the table is also in the room. The paper may be moved on the table and the same operations and properties apply to paper and table as they did apply to paper and table in Section 3.4. If the paper is removed from the table, the paper remains in the room, at least for a non-zero time (Figure 6c). The paper on the table in the room may be removed from the room by first removing it from the table and then from the room (Figure 6d). If the table is moved around the room, the paper either moves with the table and remains on it (Figure 6e) or moves off the table as result of a fast movement (Figure 6f). If the table is moved out of the room keeping the paper on it, the paper will be out of the room as well (Figure 6g).

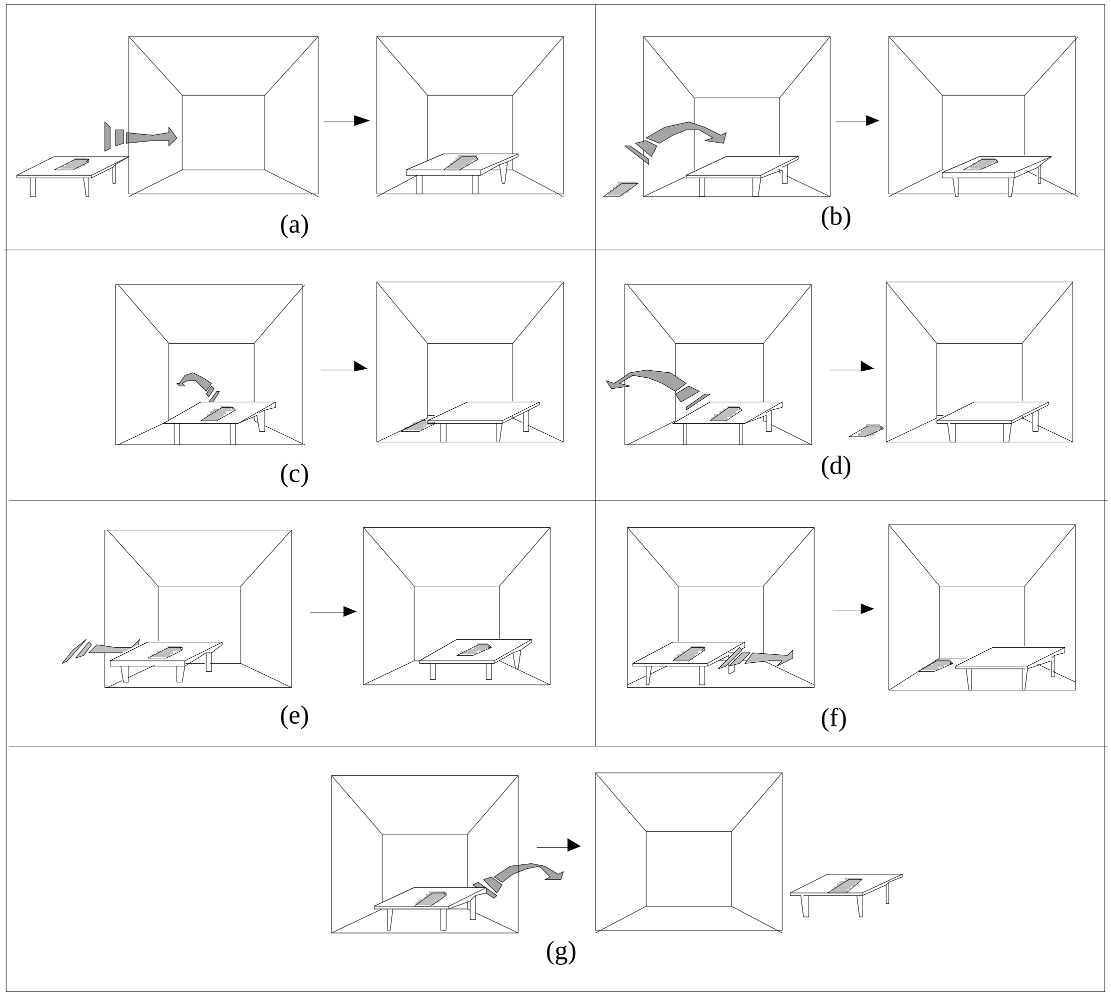


Figure 6: Paper, table, and room: (a) moving table with paper into room; (b) moving paper onto table that is in a room; (c) removing paper from table that is in room; (d) removing paper on table in room from room; (e) moving table with paper around room; (f) jerking table with paper in room; and (g) removing table with paper from room.

4. Container-Surface Algebra Derived from the Room Space

The ontology of the room space distinguishes two groups of operations and spatial relations associated with the behavior of containers and surfaces. For containers, operations in the room space are *move into*, *remove from*, and *move around*. While *move into* and *remove from* cause or change the spatial relations *in* and *out* between two objects, *move around* does not alter these spatial relations. For surfaces, operations in the room space are *move onto*, *remove from*, *move around*, and *jerk*. Similar to containers, only *move onto* and *remove from* operations define the spatial relations *on* and *off* between two objects. Although *jerk* also affects the spatial relations *on* and *off*, this operation can be defined in terms of *move onto* and *remove from* operations and therefore, it has not been considered to be a basic operation to define spatial relations associated with surfaces.

The natural-language description of this ontology was translated into an algebraic specification [17, 29] and implemented in the functional programming language Gofer [24]. Outcomes of the algebraic specification are [38]:

- Spatial relations *inside*, *outside*, *on*, and *off* are defined without ambiguities based on four basic operations: *move into*, *move onto*, and *remove from* either a container or a surface.
- The specification shows a semantic difference between surfaces and containers without using geometric properties. This difference is found as the behavior of containers and surfaces is defined within spatial configurations. The specification, therefore, is moved from the definition of objects in isolation toward the definition of spatial scenes.

Subsequently, the spatial relations derived from the room space are specified in terms of a relation algebra. This relation algebra emphasizes the inferences that can be made through composition operations.

4.1 Basic Relations

Two separate sets of basic relations are introduced by the container and surface algebras. For the container algebra, basic relations are *inside*, *outside*, *omits*, and *contains*. These four relations form pairs of converse relations. The converse relation to *inside* is *contains*, therefore, if A is *inside* B then B *contains* A. Likewise, the converse relation to *outside* is *omits*. These four relations are connected such that for different objects *outside* and *omits* represent the negation of *inside* and *contains*, respectively. If A is not *inside* a container B then A is *outside* of B. Similarly, if a container B does not *contains* A, then B *omits* A. To complete the set of basic relations for the container algebra, the relation $equal_C$ is introduced. This relation is defined between two containers or two objects that are not containers. The set of all relations corresponds to a complete and mutually exclusive set of spatial relations that are associated with containers. Thus, for a container algebra, the set of all relations {*inside*, *outside*, *contains*, *omits*, $equal_C$} defines the universal relation, $equal_C$ is the identity relation, and the subtraction of $equal_C$ from the set of all relations defines the diversity relation. The complement of a relation is then defined as the universal relation minus the particular relation. A similar relation algebra for surfaces can be defined. The universal relation of a surface algebra is {*on*, *off*, *supports*, *separated*, $equal_S$}. Unlike the container algebra, however, the relation $equal_S$ is defined between two surfaces or two objects.

To define a container-surface algebra, the two sets of relations for containers and surfaces are combined. From this new set of relations, $equal_{C\&S}$ is redefined to allow a unique identity relation for both containers and surfaces; therefore, $equal_{C\&S}$ is now defined between two objects, two containers, or two surfaces. For the container-surface algebra, the set of {*inside*, *outside*, *contains*, *omits*, *on*, *off*, *supports*, *separated*, $equal_{C\&S}$} forms the universal relation (Table 1). Consequently, the complement of each relation is also redefined. For example, while in the container algebra the complement of *contains* is {*inside*, *outside*, *omits*, $equal_C$}, in the container-surface algebra the complement of *contains* is {*inside*, *outside*, *omits*, *on*, *off*, *supports*, *separated*, $equal_{C\&S}$}.

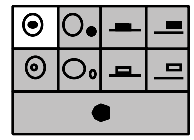
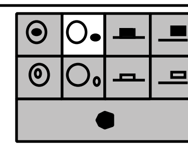
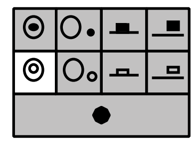
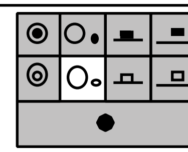
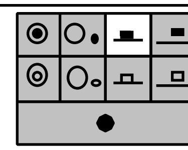
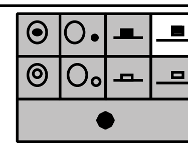
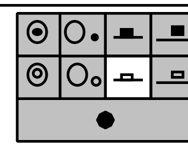
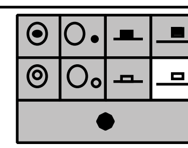
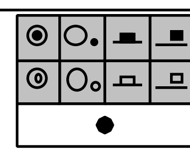
Image Schema	Relation	Result of operation	Icon for relation	Converse relation	Complement
container	A *inside* B	A moved into container B			
container	A *outside* B	A removed from container B			
container	B *contains* A	A moved into container B			
container	B *omits* A	A removed from container B			
surface	A *on* B	A moved onto surface B			
surface	A *off* B	A removed from surface B			
surface	B *supports* A	A moved onto surface B			
surface	B *separated* A	A removed from surface B			
container or surface	A $equal_{C\&S}$ B				

Table 1: Basic relations of the container-surface algebra (dark: complement relation).

4.2 Compositions

To define the composition operation, an extensive analysis over possible combinations of spatial relations within the room space was done. Following the study case, containers and surfaces were considered separately and then combined to define a composition table for the container-surface algebra. For each combination of spatial relations, we analyzed concrete cases with objects of the room space. Since spatial configurations in the room space cover different possible combinations of objects—e.g., for A *inside* B; B *contains* C, configurations in the room space exemplify cases when A and B are either objects or containers—the union of these possible compositions in the room space determines the composition *inside* ; *contains* of a generalized container algebra. Likewise, composition operations involving surfaces, and surfaces and containers in combination, were analyzed. The generalized composition tables for the container, surface, and container-surface algebras are presented in Tables 2, 3, and 4, respectively.

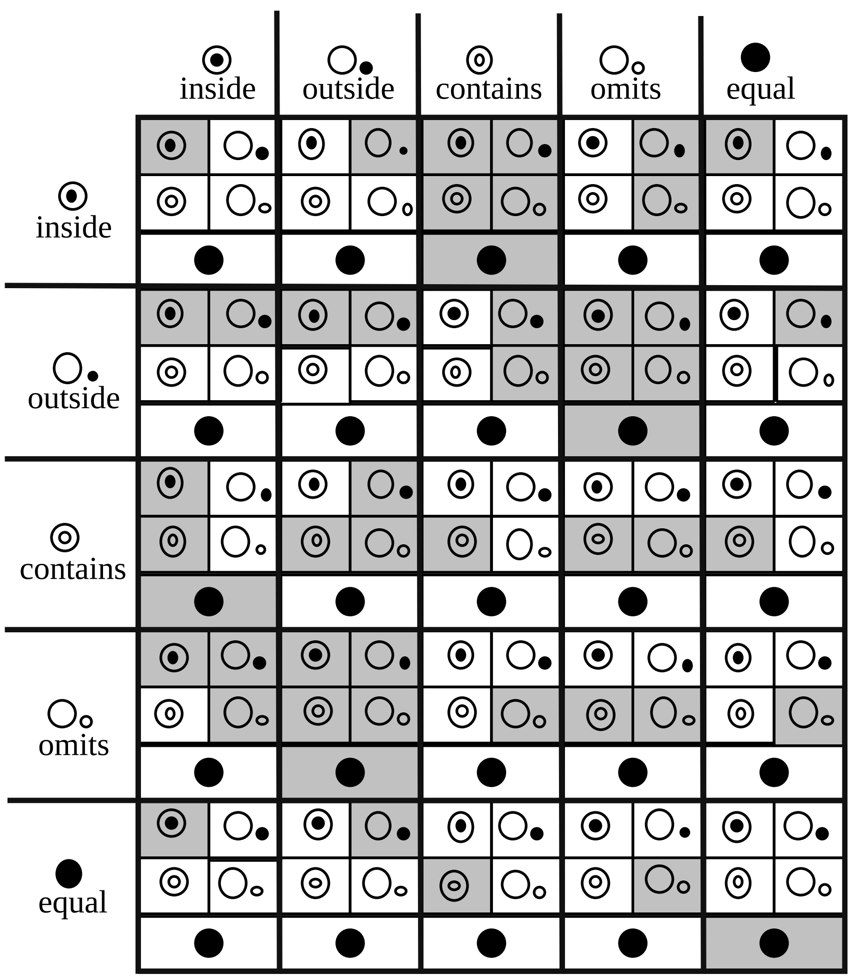


Table 2: Composition table of the container algebra (dark: possible inferred relation).

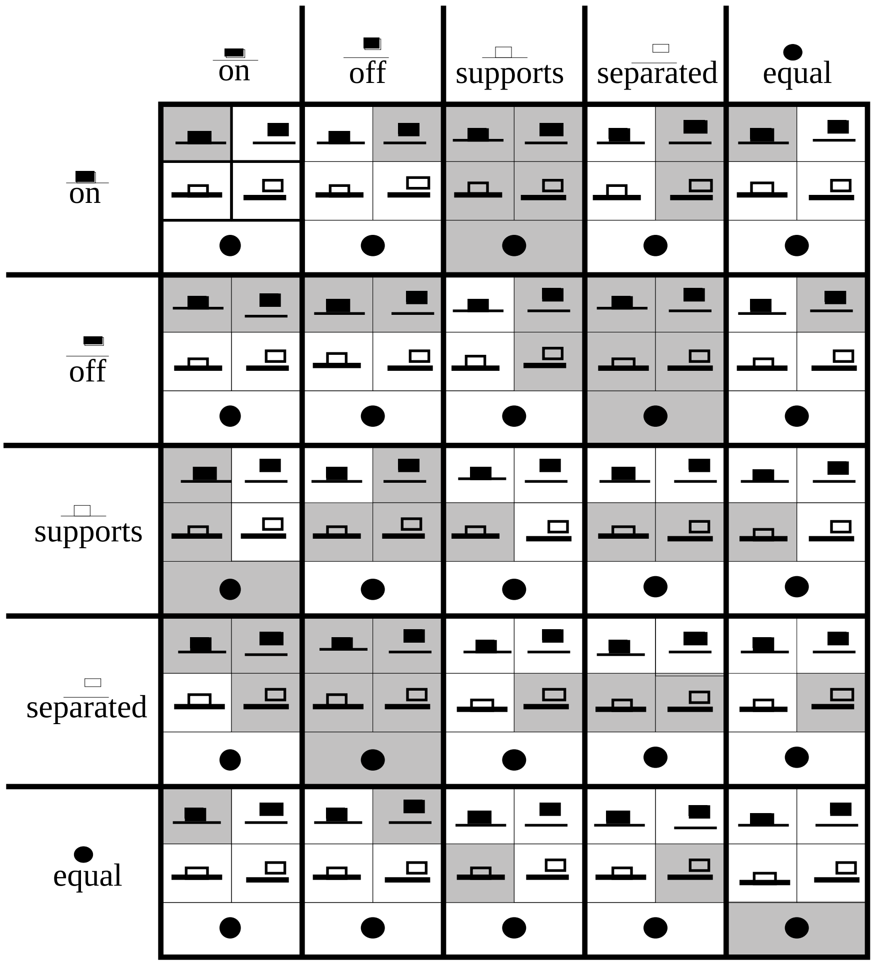


Table 3: Composition table of the surface algebra (dark: possible inferred relation).

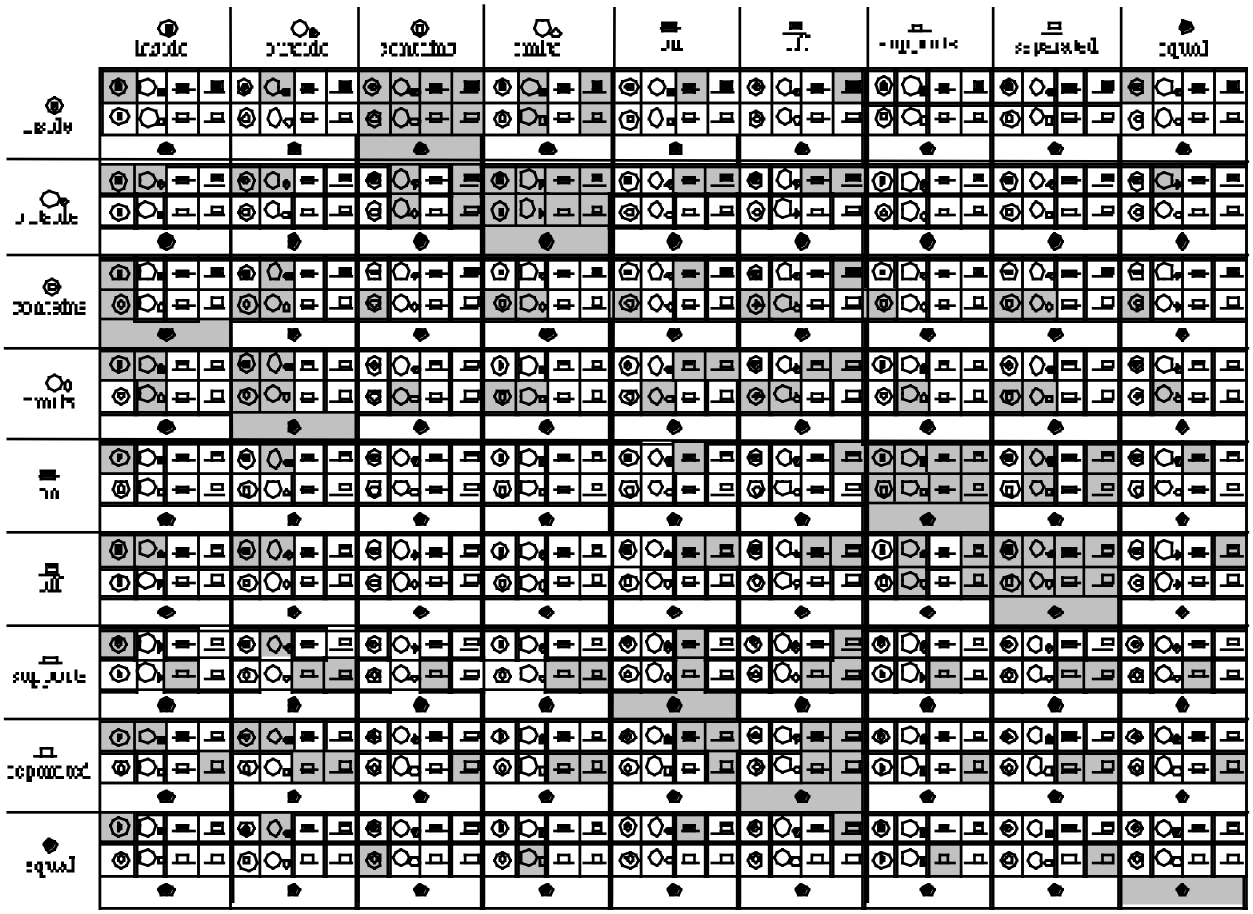


Table 4: Composition table of the container-surface algebra for objects playing only one role being either a container or surface (dark: possible inferred relation).

Three major assumptions are behind the generalized composition tables: (1) within a configuration, a physical object can play only one role, being either a container or a surface; (2) a container cannot contain an object of the same size; however, a surface can support an object of the same size; and (3) for all compositions, the relation between two objects assumes that at least one of the two objects is either a container or a surface, i.e., a composition such as a pen is in a box and a box contains a ball is not considered since our algebra does not define a spatial relation between pen and ball, which are neither containers nor surfaces.

Eight compositions are impossible (i.e., they result in the empty relation). The reason for the inability to make any inferences at all is that these compositions would require an object to switch roles. For example, if A is on B this implies that B is a surface. On the other hand, if B contains C this implies that B is a container. Composing on with contains would require B to switch roles.

All thirty-three composition operations that have a unique result carry an implicit role and relative size between objects. For example, if A is *inside* B and B is *inside* C, A must be smaller than C, C must be a container and, therefore, A is *inside* C [19]. The inverse conclusion, however, cannot be made, i.e., compositions that have an implicit role and relative size do not always result in a unique relation. For those compositions that result in the universal relation, there are no implicit constraints about the sizes and roles of the objects. For example, consider the composition A *inside* B and B *contains* C, the only constraints of this composition is that B must be a container and bigger than A and C. Between A and C, however, there is no constraint about their sizes—A may be smaller than B, larger than B, or of the same size as B.

Container and surface composition tables (Tables 2 and 3) show a strong similarity between spatial relations of containers (i.e., *inside*, *outside*, *contains*, and *omits*) and spatial relations of surfaces (i.e., *on*, *off*, *supports*, and *separated*). The composition table for surfaces can be converted into the composition table for containers as the spatial relations *on*, *off*, *support*, and *separated* are replaced by the spatial relations *in*, *out*, *contains*, and *omits*, respectively. In algebraic terms, the container and surface algebras are isomorphic [29] since it is possible to define a function **f** that maps relations of containers onto relations of surfaces such that the container and surface algebras are equivalent, using $;_C$ and $;_S$ as the composition of the container and surface algebras, respectively, and c_1 and c_2 are basic relations of the container algebra (Equation 4).

$$\mathbf{f}\,(c_1 \,;_C\, c_2) = (\mathbf{f}\,(c_1) \,;_S\, \mathbf{f}(c_2)) \tag{4}$$

Likewise, the composition table for the container-surface algebra reflects a similarity between the inferences that combine containers and surfaces. In general terms, composition a ; b, where a and b are spatial relations associated with image schemata (either a container or a surface) is_a and is_b, respectively, results in a set of spatial relations that is equivalent to the composition b ; a as spatial relations *inside*, *outside*, *contains*, and *omits* are substituted by spatial relations *on*, *off*, *supports*, and *separated*, respectively.

The definition of the container-surface algebra is weakly typed. For example, the relation *A contains B* only restricts *A* to be a container. An alternative approach to the definition of the container-surface algebra is the use of typed relations. Under this approach the relation *A contains B* is decomposed into *A* $contains_{cc}$ *C*, *A* $contains_{cs}$ *S*, and *A* $contains_{co}$*O*, where *C* is a container, *S* is a surface, and *O* is neither a container nor a surface. Consequently, composition operations are defined over objects of the same type. Although a typed definition of the container-surface algebra is a valid approach, we follow a top-down approach and favor a less detailed definition of the container-surface algebra, since we pursue a definition of spatial relations that is cognitively sensible to human reasoning.

4.3 Properties of the Container-Surface Algebra

The properties of a relation algebra (Equation 1a-g) were exhaustively examined for the container-surface algebra by using a program written in C++. The analysis reveals that the container algebra and the surface algebra fulfill all seven axioms, whereas the integrated container-surface algebra violates one axiom. The associative property that applies to the container and surface algebras is no longer applicable for the integrated container-surface algebra. We checked all possible combinations of composition operations and we obtained that 13% of these combinations are not associative; therefore, the container-surface algebra may draw different conclusions from two different reasoning paths, where one of them produces a subset of the possible spatial relations derived from the other one. For instance, the composition operation (*inside* ; *contains*) ; *inside* results in the set {*inside*, *outside*, *contains*, *omits*, *supports*, *separated*, $equal_{C\&S}$}, whereas the composition operation *inside* ; (*contains* ; *inside*) results in the universal relation. Although the container-surface algebra is not associative, it is semiassociative (Equation 3a).

5. Variations of the Container-Surface Algebra

By forcing objects in our room space to have only one role, we limit the number of scenarios where the container-surface algebra may be applied even if we use the same type of objects such as boxes. Therefore, a modification of the container-surface algebra derived has been explored by allowing objects to be containers or surfaces at the same time. For example, we can consider now a closed box that has a pen on the top and another smaller box inside. Under this assumption, a new composition table is derived (Table 5). For this new table, since a container can also be seen as a surface, the number of possible spatial relations product of a composition operation increases and therefore, the reasoning power is reduced around 50%. For example, whereas the composition *A inside B*; *B inside C* results in one inferred relation (*A inside C*) when objects play only one role, two relations (*A inside C* or *A off C*) are product of the same composition when objects can play more than one role. Like in the case where objects play only one role at a time, this variation of the container-surface algebra holds all properties of a relation algebra except the associativity.

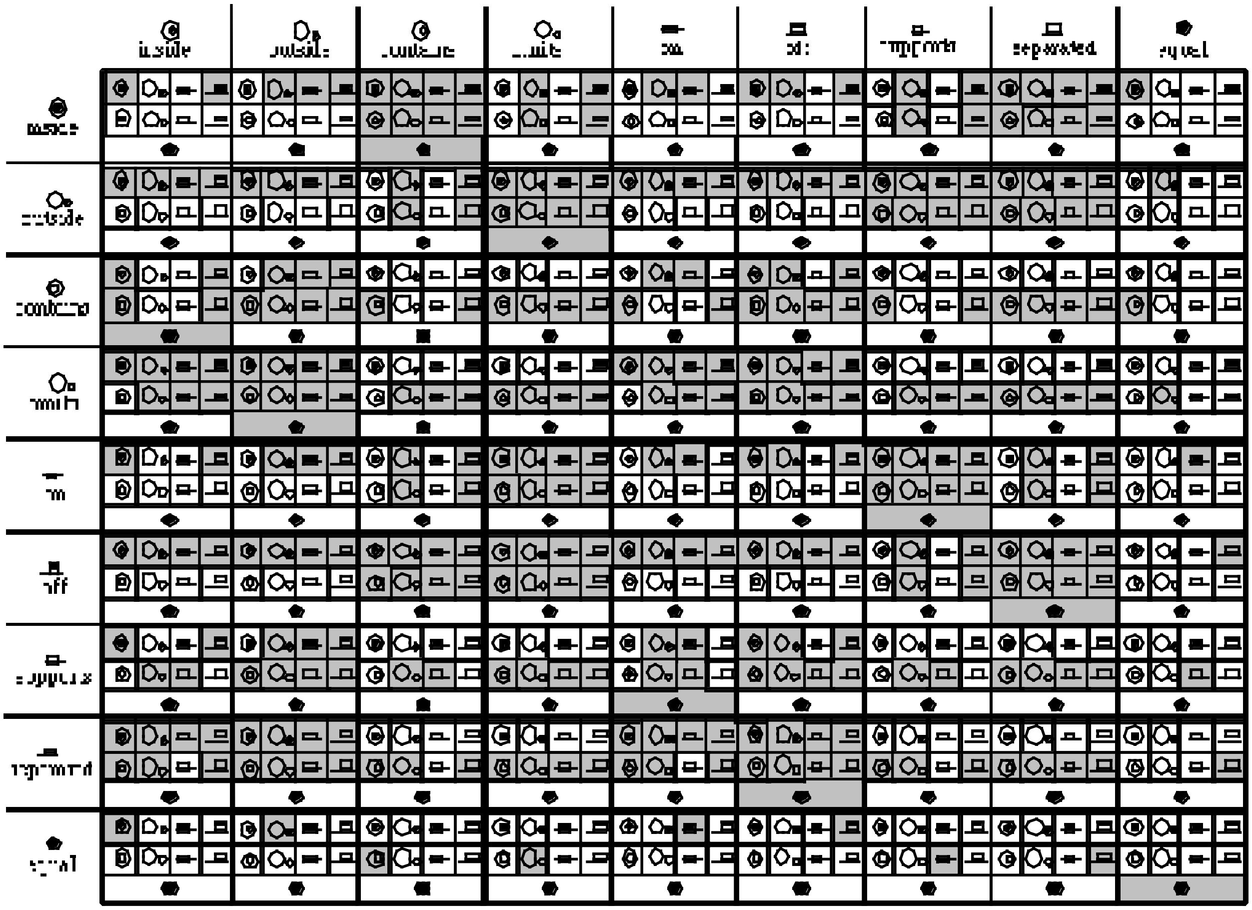


Table 5: Composition table of the container-surface algebra where objects can be simultaneously containers and surfaces (dark: possible inferred relation).

Compositions that have an implicit role and relative size are more likely to produce unique results—thirty-three vs. nine cases, all of which result from compositions with *equal*$_{C\&S}$. Hence, it is of interest to explore how much we can gain by constraining the relative size and role of objects for those compositions that do not have implicit constraints. For example, consider the modified composition table for the container-surface algebra (Table 5) and the composition *inside* ; *contains* that gives as result the universal relations (i.e., no information can be derived from the composition). When the relative size of the objects is known the set of possible spatial relations for each composition operation is much smaller (Table 6). The relative size of objects is a scalar magnitude that is represented qualitatively by partial orders—*A* is larger than *C*, *A* is the same size as *C*, or A is smaller than C [19]. For the entire composition table of the container-surface algebra, the improvement of the precision of the inferences is 18% when the relative size of objects is constrained. This improvement is calculated as the mean of the percentage of the difference between the number of spatial relations in the general table and the number of spatial relations after applying a refinement.

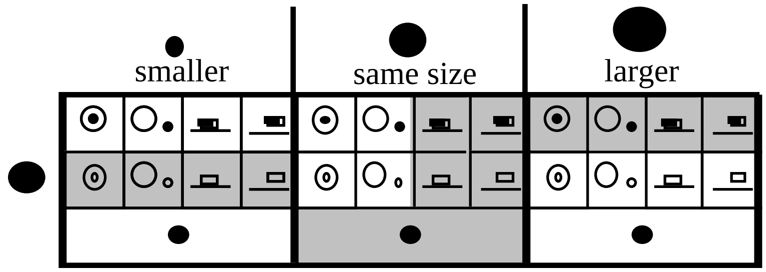


Table 6: Composition *inside* ; *contains*: refinement by size (dark: possible inferred relation).

Likewise, if the roles of the objects are known—A and C may be containers, surfaces, or neither—the set of possible spatial relations may be reduced as well. Table 7 shows the nine refinements due to considering the roles of A and C in the composition A *inside* B and B *contains* C. For the entire composition table of the container-surface algebra, the improvement of the precision of the inferences is 43% when the roles of the objects are known.

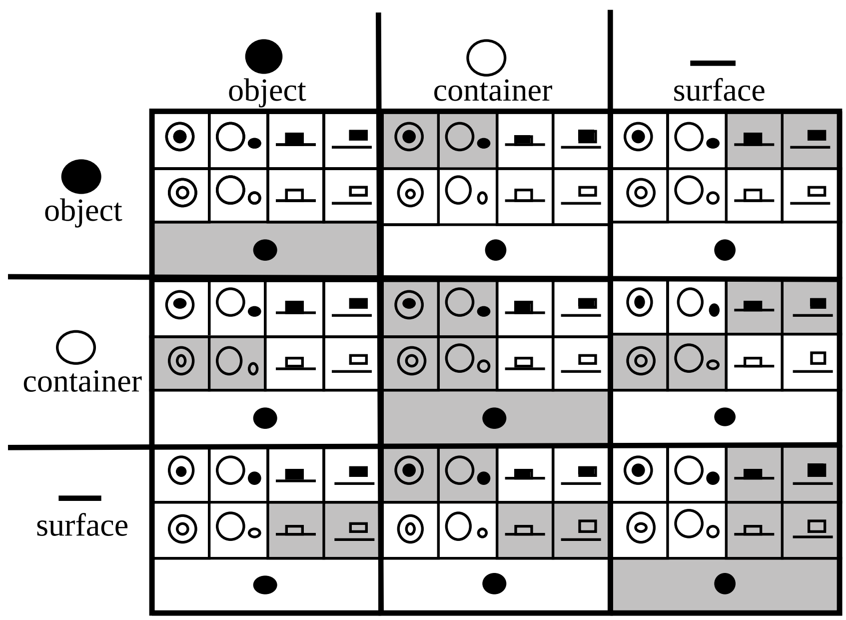


Table 7: Composition *inside* ; *contains*: refinement by role (dark: possible inferred relation).

Finally, if both roles and relative sizes are known, more detailed information can be extract from the composition table. Table 8 shows the refined inferences for *inside* ; *contains*. The analysis over the entire composition table shows 49% crisper inferences when both role and relative size are considered together. For each composition, the union of all possible spatial relations of the refinements is equal to the composition given in the general Table 5.

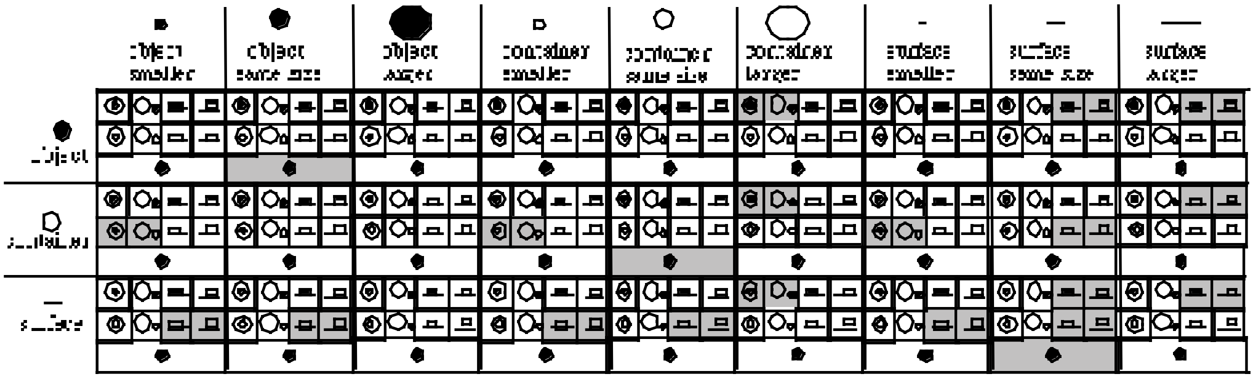


Table 8: Composition *inside* ; *contains*: refinement by role and size (dark: possible inferred relation).

Similarly, it is possible to refine the composition table that considers only one role for objects (Table 4). The improvement in this case is smaller than the improvement for the modified composition table (Table 5): 14% for relative size, 6% for role, and 16% for relative size and role. For this composition table the constraint by role does not provide a significant improvement, since objects can play only one role and the implicit role of a composition operation by itself imposes a constraint.

6. Conclusions and Future Work

A new approach to define spatial relations based on operations upon containers and surfaces was investigated. The behavior of the relations was derived from a case study of a room space. A relation algebra was defined based on nine spatial relations: *inside*, *outside*, *contains*, *omits*, *on*, *off*, *supports*, *separated*, and *equal*. This container-surface algebra holds all properties of Tarski's relation algebra, except for the associativy.

A greater reasoning power of the composition operations can be achieved as relative sizes (smaller, equal, and larger) and specific roles (container, surface, or unspecified object) of the objects are considered. Compositions with objects that may play more than one role represent less powerful mechanisms of inference. A further analysis, however, can reduce possible inferred relations by considering that people may use image schemata in a prioritized way or based on a dominating object.

The behavior of containers and surfaces is more generally applicable to other scenarios than to the room space [38]. For example, if a forest is on a mountain and the mountain is on a peninsula, we can conclude that the forest is also on the peninsula. However, there are cases in which the container-surface inferences may appear counter-intuitive. For instance, if a paper is on a table and the table is on the floor, is the paper on the floor? Work is in progress to evaluate whether all spatial inferences obtained from the room space may be applied to scenarios with larger-sized objects or mixed-sized objects.

An interesting open question is the mapping from the spatial-relation symbols of the relation algebra onto natural language, and vice-versa. The term *on*, for instance, stands for information about a surface ("on the table") as well as closeness ("on the beach"). There are also cross-linguistics differences ¾ for example, while in English "on the bus" refers to a surface, in German "im Bus" is associated with a container. It was hypothesized that these linguistics differences may indicate different spatial conceptualizations [30], and a future study with the inferences of the container-surface algebra may provide supporting evidence.

7. Acknowledgments

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