# Part and Complement: Fundamental Concepts in Spatial Relations

John G. Stell

School of Computing, University of Leeds,
Leeds, LS2 9JT, U. K.
jgs@comp.leeds.ac.uk

**Abstract.** The spatial world consists of regions and relationships between regions. Examples of such relationships are that two regions are disjoint or that one is a proper part of the other. The formal specification of spatial relations is an important part of any formal ontology used in qualitative spatial reasoning or geographical information systems. Various schemes of relationships have been proposed and basic schemes have been extended to deal with vague regions, coarse regions, regions varying over time, and so on. The principal aim of this paper is not to propose further schemes, but to provide a uniform framework within which several existing schemes can be understood, and upon which further schemes can be constructed in a principled manner. This framework is based on the fundamental concepts of part and of complement. By varying these concepts, for example allowing a part-of relation taking values in a lattice of truth values beyond the two-valued Boolean case, we obtain a family of schemes of spatial relations. The viability of this approach to spatial relations as parameterized by the concepts of part and complement is demonstrated by showing how it encompasses the RCC5 and RCC8 schemes as well as the case of 'egg-yolk regions'.

## 1 Introduction

### 1.1 Spatial Relations

The spatial world consists of regions – occupied by particular countries, seas, or mountain ranges, regions where specific philosophies, ontologies, or poisonous fungi flourish. These regions are not separate things, unrelated to each other: some regions are parts of others, regions may be disjoint, regions may touch only at their boundaries. More elaborate relationships than these examples exist: one region may be disjoint from another but be contained within its convex hull; two regions may vary over time and at all times the regions overlap but neither is ever a proper part of the other.

Various systems of regions and formalized relations between them have been used in qualitative spatial reasoning and in geographical information systems (GIS). Some references to work in this area can be found in section 1.2 below. These systems can be used as foundations for formal ontologies for geographic, and more generally, spatial, entities. That is, we can use formalizations of regions and their relationships to construct formal descriptions of objects in the geographic world.

This paper does not describe further formal schemes of spatial relations, although the technique it introduces can be used to do this. Instead it shows how some of the existing schemes can be derived in a uniform way from the concepts of part and complement alone. The central idea is that the relationship of a region, $A$, to a region, $B$, can be measured by a triple of values:

$$( A \text{ is part of } B,\ A \text{ is part of the complement of } B,\ B \text{ is part of } A\ ).$$

The Region Connection Calculus (RCC) [CB+97,Ste00a] provides an axiomatization of regions, and taking $A$ and $B$ to be regions in this sense and interpreting part-of and complement as in RCC, this triple gives us three Boolean values. For example, ( true, false, true ) in the case that $A$ and $B$ are equal. The classification arising from the different triples of Boolean values is an extension to the usual RCC5 system in that empty regions are permitted. This is described in detail in section 2 below.

The significance of the new approach, based on the triple of values, is that it can easily be used to provide systems of spatial relations in contexts other than the crisp regions axiomatized in RCC. Examples of such contexts include vague regions, regions varying over time, regions which are both vague and vary over time, coarse regions and so on. Different systems of spatial relations arise by taking different notions of part and complement. The main message of the paper is thus:

> In order to construct a system of spatial relations for a particular kind of region, it is only necessary to provide appropriate notions of part and complement for that kind of region.

The part-of relation need not be Boolean valued, and cases where it is three-valued and six-valued appear in the paper. The first three-valued example appears in section 3 below, which shows how RCC8 fits into our framework. Sections 4 and 5 apply the framework to vague regions. Definitions of part for vague regions are presented in section 4 and the systems of spatial relations which result are discussed in section 5.

### 1.2 Previous Work on Spatial Relations

There are two principal families of spatial relations. These are the 'intersection models' developed by Egenhofer et al., and the schemes based on the Region Connection Calculus developed by Cohn et al. In each case there are two basic schemes, one of which subsumes the other.

Given spatial regions $A$ and $B$, with interiors $A^\circ$, $B^\circ$, and boundaries $\partial A$, $\partial B$ respectively, the 4-intersection model [EF91] measures how $A$ relates to $B$ by noting which of the intersections in the sequence ($\partial A \cap \partial B$, $A^\circ \cap B^\circ$, $\partial A \cap$

$B^\circ$, $A^\circ \cap \partial B$) are empty and which are non-empty. The 9-intersection model, introduced in [EH91], is an extension of this which uses intersections involving complements of regions as well as interiors and boundaries.

The two schemes based on the Region-Connection Calculus are known as RCC5 and RCC8, as they distinguish five and eight relationships respectively. Unlike the 4- and 9-intersection models, which take regions to be sets of points with topological structure, the RCC schemes are based on an axiomatization of regions as primitives, not constructed from points. Descriptions of RCC5 and RCC8, with references to the papers where they first appeared, can be found in [CB+97].

These schemes have been extended in various ways to allow for more sophisticated distinctions in how regions relate to each other. Finer distinctions between relationships of regions than are provided by the 4-intersection model can be obtained by counting the number of connected components in the intersection of $A$ with $B$, and in certain other regions determined by $A$ and $B$. The investigation of this use of connected components is due to Galton [Gal98]. Another way of extending the 4-intersection model, which provides for still finer distinctions than in Galton's technique, is described in [EF95]. The RCC schemes have been extended to systems such as RCC15 and RCC23 [CB+97], which take account of the convex hulls of the regions involved.

Düntsch et al. [DWM01] have shown how an analysis of RCC5 and RCC8 from the viewpoint of relation algebras leads to extensions which they call RCC7 and RCC10. These include, for example, making distinctions between regions being disjoint with their union being the whole space, and disjointness without the union being the whole space.

Although many researchers have assumed that reasoning with systems of qualitative relations would be related to how humans actually reason, there have been relatively few empirical studies of this issue. The cognitive adequacy of sets of spatial relations has been investigated by Renz and others in [RRK00,KRR97]. Other work by Renz and Nebel addresses the efficiency of reasoning with spatial relations [RN98], and Renz' recent book [Ren02] discusses this as well as other related work.

Another direction for extension is to vague regions. Two approaches based on the 9-intersection model, are the work of Clementini and Di Felice [CF97] on regions with 'broad boundaries', and the work of Zhan [Zha98] using fuzzy regions. The extension of the RCC schemes to accommodate vague regions has been addressed by Lehmann and Cohn [LC94], and by Cohn and Gotts [CG96].

## 2 RCC5 and its extension to permit empty regions

### 2.1 The RCC5 Scheme

If $A$ and $B$ are regions satisfying the axioms of the Region-Connection calculus, the relationship of $A$ to $B$ can be classified as one of the following possibilities.

- $A$ is disjoint from $B$.

– $A$ and $B$ overlap but neither is a part of the other.
– $A$ is inside $B$, i.e. $A$ is a proper part of $B$.
– $A$ contains $B$, i.e. $B$ is a proper part of $A$.
– $A$ and $B$ are equal.

These five cases form a jointly exhaustive and pairwise disjoint (JEPD) set of relations. Thus, for any given regions $A$ and $B$, exactly one of the five cases holds. The RCC5 relationship between $A$ and $B$ can be defined using only the part-of relation, $\leq$, and the complement operation, $\neg$. Consider the triple of Boolean values

$$( A \leq B,\ A \leq \neg B,\ B \leq A ).$$

Knowing these three Boolean values allows us to determine the relationship between $A$ and $B$, as shown in table 1. The first five lines of the table give the usual RCC5 possibilities.

### 2.2 Extended version of RCC5

A triple of Boolean values gives eight cases in total, but RCC5 only distinguishes five cases. The discrepancy is due to the assumption in the RCC theory that regions must be non-empty. This insistence that regions be non-empty does seem to be widely accepted in work on qualitative spatial reasoning. One reason for this position has been advanced by Bennett [Ben95, section 3.5].

> "If null-regions are allowed, they have properties which may seem counter-intuitive (for example the null region is both part of and disconnected from any other region) and many useful and apparently sound inferences may not hold if it is allowed that one of the regions involved may be null."

However, it is easy to find practical situations in which a query to a spatial database might be expected to return the empty region. For example one might want to select part of a region $A$ having some property, where in fact the property is not satisfied anywhere in $A$. Thus are certainly pragmatic reasons for allowing the empty region. However, there are certainly situations where the empty region does need to be treated differently from other regions. What is being proposed here is not that the empty region can be handled in the same way as any other region, but that it is not appropriate to exclude it to the extent which sometimes appears to be thought necessary.

The objection that allowing the empty region causes problems, because, for example, it is both part of and disconnected from any other region, can be overcome. It is true that if we keep the RCC5 scheme and allow the empty region, we cannot avoid losing the mutually exclusive property. The solution to this difficulty is not to proscribe the empty region, but to extend the five cases distinguished by RCC5. This extension is that set out in table 1, and will be referred to as RCC5$^+$. The relationship between two non-empty regions in RCC5$^+$ is exactly the same as in the RCC5 case.

| | | $A \leq B$ | $A \leq \neg B$ | $B \leq A$ |
|---|---|---|---|---|
| $A$ and $B$ are disjoint and non-empty | $\mathsf{DC}\,(A, B)$ | false | true | false |
| $A$ and $B$ share a non-empty proper part | $\mathsf{PO}\,(A, B)$ | false | false | false |
| $A$ is a non-empty proper part of $B$ | $\mathsf{PP}\,(A, B)$ | true | false | false |
| $A$ has $B$ as a non-empty proper part | $\mathsf{PPi}\,(A, B)$ | false | false | true |
| $A$ and $B$ are equal and non-empty | $\mathsf{EQ}\,(A, B)$ | true | false | true |
| $A$ and $B$ are equal and empty | $\varnothing\mathsf{EQ}\,(A, B)$ | true | true | true |
| $A$ is the empty proper part of $B$ | $\varnothing\mathsf{PP}\,(A, B)$ | true | true | false |
| $B$ is the empty proper part of $A$ | $\varnothing\mathsf{PPi}\,(A, B)$ | false | true | true |

**Table 1.** Extended RCC5 using a two-valued part-of relation

## 3 An extended version of RCC8

The basic scheme $(A \leq B, A \leq \neg B, B \leq A)$ can be applied to spatial regions, as in its application to RCC5, but can be used, more generally, for arbitrary sets $A$ and $B$, since no topological structure is assumed on $A$ and $B$. To deal with RCC8 we need a more elaborate notion of part. Instead of the statement '$A$ is a part of $B$' being either true or false, we allow it to take one of three truth values. To do this we have to assume regions have interiors. Denoting the interior of $A$ by $A^\circ$, the three-valued relation part-of relation is defined:

$$A \preccurlyeq B = \begin{cases} \mathsf{T} & \text{if} \quad A \leq B^\circ \\ \mathsf{M} & \text{if} \quad A \not\leq B^\circ \quad \text{and} \quad A \leq B \\ \mathsf{F} & \text{if} \quad A \not\leq B \end{cases}$$

Assuming regions to be regular closed, we define the complement, $\neg B$, to be the set-theoretic complement of the interior, so $\neg B = (B^\circ)'$. Using the fact that $(\neg B)^\circ = B'$, we get the following

$$A \preccurlyeq \neg B = \begin{cases} \mathsf{T} & \text{if} \quad A \leq B' \\ \mathsf{M} & \text{if} \quad A \not\leq B' \quad \text{and} \quad A \leq (B^\circ)' \\ \mathsf{F} & \text{if} \quad A \not\leq (B^\circ)' \end{cases}$$

The triple of values $(A \preccurlyeq B, A \preccurlyeq \neg B, B \preccurlyeq A)$ allows for $3^3 = 27$ possibilities, but only 11 of these can actually occur. For example, it is impossible that $A \preccurlyeq B = \mathsf{T}$ and $A \preccurlyeq \neg B = \mathsf{M}$. For this would imply $A \leq B^\circ$ and $A \leq (B^\circ)'$, so that $A$ is the empty region. But then we have to have $A \preccurlyeq \neg B = \mathsf{T}$ contrary to the original assumption. A detailed analysis of the various cases reveals that only 11 cases occur, and these correspond exactly to the usual RCC8 cases together with three extra cases. These three cases allow that both $A$ and $B$ are empty ($\varnothing\mathsf{EQ}$), that $A$ is the empty proper part of $B$ ($\varnothing\mathsf{PP}$), and that $B$ is the empty proper part of $A$ ($\varnothing\mathsf{PPi}$) The 11 cases are set out in table 2.

## 4 Vague Regions

We have seen in sections 2 and 3 how the RCC5 and RCC8 relations between two crisp regions can be expressed using appropriate concepts of part and com-

| | DC | EC | PO | TPP | TPPi | NTPP | NTPPi | EQ | ∅EQ | ∅PP | ∅PPi |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $A \preccurlyeq B$ | F | F | F | M | F | T | F | T | T | T | F |
| $A \preccurlyeq \neg B$ | T | M | F | F | F | F | F | F | T | T | T |
| $B \preccurlyeq A$ | F | F | F | F | M | F | T | T | T | F | T |

**Table 2.** Extended RCC8 using a three-valued part-of relation

plement. This section investigates how these concepts can be extended to vague regions. In particular, two possible notions of part for vague regions are introduced, one taking values in a set of three truth values, and the other being a six-valued relation. This six-valued relation is used in section 5 to provide a simple way of extending RCC5$^+$ to vague regions.

### 4.1 A Semantic Approach to Vague Regions

The application of the triple-based classification scheme to vague regions can be carried out without any commitment to a specific definition of vague region. All that is assumed is that associated to the set of vague regions $R^V$ is a set of crisp regions $R$, and that the purpose of a vague region is to describe a set of crisp regions. Thus each vague region $v$, has an associated set of possible crispings, $[\![v]\!]$. The crisp regions $R$ need to support a partial order, $\leq$, and a complement operation $\neg : R \to R$, and there are greatest and least elements $\top$ and $\bot$ respectively with respect to $\leq$. The complement operation is assumed to satisfy the equation $\neg\neg r = r$, and to be order reversing in the sense that $r \leq s$ implies $\neg s \leq \neg r$. These assumptions are easily seen to be satisfied in the case that $R - \{\bot\}$ is a set of regions satisfying the axioms of the region-connection calculus, but there is no need to assume that $R$ has this particular form.

In making statements about vague regions, we often need to say that certain things hold under all crispings of a regions, or under some crisping. Thus we make use of logical symbols $\Box_v$ and $\Diamond_v$, with the following definitions, where $\varphi(v)$ is any formula containing the free variable $v$.

$$\Box_v \varphi(v) = \forall x \in [\![v]\!]\ \varphi(x) \quad \text{and} \quad \Diamond_v \varphi(v) = \exists x \in [\![v]\!]\ \varphi(x)$$

This approach in which a vague region $v \in R^V$ has an associated semantic interpretation $[\![v]\!] \subseteq R$ allows us to deal simultaneously with several different proposal for vague regions. For example, taking $R - \{\bot\}$ to be a model of the region-connection calculus, and defining $R^V = \{\langle r, s\rangle \in R \times R \mid r \leq s\}$ we have the 'egg-yolk' regions described in [LC94,CG96]. In this case one possible semantics is $[\![\langle r, s\rangle]\!] = \{x \in R \mid r \leq x \leq s\}$. However, some notions of vague, or uncertain, region for which this semantics is not suitable, are easily accommodated within the framework. For instance, the boundaries of a parcel of land may be disputed between two parties. In this case the disputed land is a vague region, $v$, where $[\![v]\!] = \{r_1, r_2\}$ is a set consisting of the two interpretations held by the two sides to the dispute. Although an egg-yolk region, $\langle r, s\rangle$, can easily be constructed, by taking $r = r_1 \cap r_2$, and $s = r_1 \cup r_2$, the above semantics for $\langle r, s \rangle$ is inappropriate in this context.

Cohn and Gotts [CG96, p178] emphasize that they do not associate any particular permitted set of crispings with an egg-yolk region: "We do not and need not specify *exactly* where the limits of acceptability lie". In the present paper it is assumed that a set of possible crispings is given, but it would be possible to extend the framework by replacing the set of crispings by a fuzzy set of some kind. This would allow some potential crispings to be unequivocally crispings whereas others would have some degree of doubt associated with them.

### 4.2 Part-of Relations for Vague Regions

A simple three-valued part of relation can be defined for vague regions. Using the notation $\leq_3$ for this relation, the three possible values of $u \leq_3 v$ correspond to the situations that $u$ is part of $v$ no matter how either is crisped, that $u$ is a part of $v$ under some crispings but not others, and that $u$ is never a part of $v$ under any crisping. The relation is defined as follows.

$$u \leq_3 v = \begin{cases} \mathsf{T} \ \text{(true)} & \text{if} \quad \Box_u \Box_v u \leq v \\ \mathsf{M} \ \text{(maybe)} & \text{if} \quad \Diamond_u \Diamond_v u \leq v \text{ and } \Diamond_u \Diamond_v u \not\leq v \\ \mathsf{F} \ \text{(false)} & \text{if} \quad \Box_u \Box_v u \not\leq v \end{cases}$$

A more sophisticated measure of the extent to which $u$ is a part of $v$ uses the formulæ $\Box_u \Diamond_v u \leq v$ and $\Diamond_u \Box_v u \leq v$. These express the statement that for every crisping of $u$, it is possible to find a crisping of $v$ such that $u \leq v$, and the statement that there is some crisping of $u$ such that however $v$ is crisped, $u \leq v$. These formulæ are related to two others appearing in the definition of $\leq_3$ by the lattice of implications shown in figure 1. We can then measure the extent to

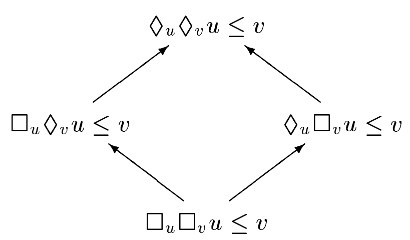


**Fig. 1.** Implications between facts about crispings

which $u$ is a part of $v$ by noting which of these four formulæ are true and which false. Because of the implications between the four, only six subsets of these are possible. The six subsets correspond to the upper sets in the lattice of four formulæ above. This leads to the lattice of six mutually exclusive and exhaustive possibilities in figure 2. Note that for any formula $P(x)$, $\neg\Box_x P(x) = \Diamond_x \neg P(x)$ and $\neg\Diamond_x P(x) = \Box_x \neg P(x)$. Thus, for example, in the diagram $\Box_u \Diamond_v u \not\leq v$ is equivalent to $\neg\Diamond_u \Box_v u \leq v$.

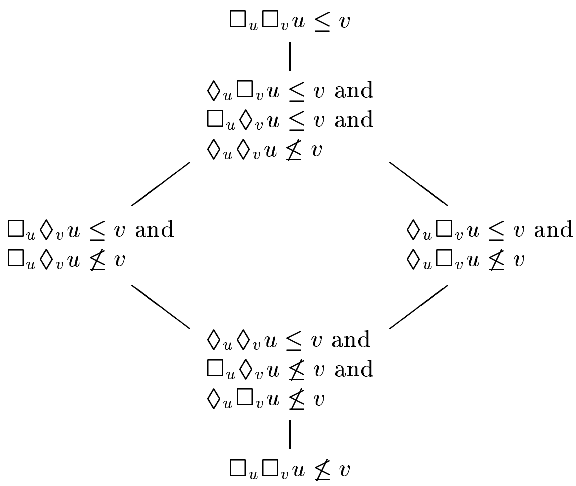


**Fig. 2.** Lattice of the six truth values

It is convenient to have a notation for the six truth values, and the diagram below shows the one used here. Each of the six values corresponds to a subset of the lattice of four values in figure 1, so Top corresponds to the top element of the four holding; Left, and Right are the left and right elements respectively; Both is both left and right; and All and None are all the elements and none of them respectively.

The notation $\preccurlyeq$ will be used for the six-valued part-of relation, so the expression $u \preccurlyeq v$ will take one of the six values in figure 3.

## 5 Relating Vague Regions

To use the scheme

( $A$ is part of $B$, $A$ is part of the complement of $B$, $B$ is part of $A$ )

as a way of classifying relationships between vague regions, we can use either the three-valued relation, $\leq_3$, or the six-valued relation, $\preccurlyeq$, as the notion of part. In either case the appropriate notion of complement is given by defining $\neg v$ to be the vague region with semantics $[\![\neg v]\!] = \{\neg x \in R \mid x \in [\![v]\!]\}$.

If we classify the relationship of $u$ to $v$ by the triple $(u \leq_3 v, u \leq_3 \neg v, v \leq_3 u)$, there are 27 cases. All of the 27 cases can arise, and in the majority of cases

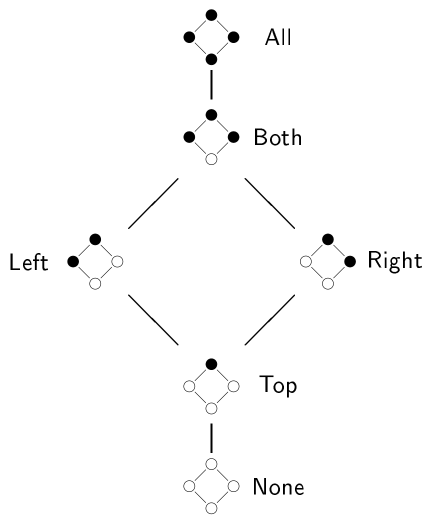


**Fig. 3.** Notation for the six truth values

specific inferences may be drawn about which RCC5$^+$ relationships hold under possible crispings. For example, the triple (M, F, F), corresponds to the fact that under some crispings $u$ and $v$ overlap, and under other crispings $u$ is a proper part of $v$, but no other RCC5$^+$ relationships are possible under any crispings. Not all the 27 triples yield information as specific as this. In cases where two or three of the components of the triple take value M, we cannot be certain exactly which RCC5$^+$ relationships are possible.

If we classify the relationship of $u$ to $v$ by the triple $(u \preccurlyeq v, u \preccurlyeq \neg v, v \preccurlyeq u)$, there are $6^3 = 216$ potential cases. Some of these 216 cases are impossible without any further assumptions about the nature of vague regions or their crispings; other restrictions arise once additional assumptions are made. In the remainder of this section these restrictions are analysed. The discussion is divided into two parts; first we examine restrictions that hold under any notion of crisping, and secondly the restrictions which arise from assuming that the set of all crispings is closed under intersections.

### 5.1 General Restrictions

We first consider the restrictions on the 216 potential values of a triple $(u \preccurlyeq v, u \preccurlyeq \neg v, v \preccurlyeq u)$ which do not make assumptions about the structure of $[\![u]\!]$ and $[\![v]\!]$.

**Lemma 1** *If* $\Box_u \Box_v u \leq v$ *and* $\Box_v \Diamond_u v \leq u$ *and* $\Diamond_v \Box_u v \leq u$ *then* $\Box_v \Box_u v \leq u$.

**Proof** From $\Box_u \Box_v u \leq v$ and $\Box_v \Diamond_u v \leq u$, it follows that $\Box_v \Diamond_u v = u$. This means that every crisping of $v$ is also a crisping of $u$, i.e. that $[\![v]\!] \subseteq [\![u]\!]$.

From $\Box_u \Box_v u \leq v$ and $\Diamond_v \Box_u v \leq u$, it follows that $\Diamond_v \Box_u v = u$. This means that some crisping of $v$ is equal to every crisping of $u$, so $u$ must be a crisp region. But, since $[\![v]\!] \subseteq [\![u]\!]$, we deduce that both $u$ and $v$ are crisp regions with the same unique crisping. In particular, $\Box_v \Box_u v \leq u$.

As an immediate consequence we have:

**Lemma 2** *If the relationship between regions $u$ and $v$ is $(a, b, c)$, then neither of the following combinations for $a$ and $c$ is possible.*

1. *$a =$ All and $c =$ Both,*
2. *$a =$ Both and $c =$ All.*

It is convenient to arrange the remaining restrictions by fixing the middle component $u \preccurlyeq \neg v$ and considering what restrictions this places on the other two components. The results are given in the following lemmas, each dealing with one possible value of the middle component for which restrictions exist.

**Lemma 3** *If $u \preccurlyeq \neg v =$ Right, then $\Diamond_v \Box_u v \leq u$ is impossible, and hence $v \preccurlyeq u$ cannot be Right or Both or All.*

**Proof** As $\Diamond_u \Box_v u \leq \neg v$, some crisping of $u$ is outside every crisping of $v$. Now if $\Diamond_v \Box_u v \leq u$, then also some crisping of $v$ is inside every crisping of $u$. This can only be if $v$ admits the empty region as a cripsing. But this would contradict our assumption that $\Diamond_u \Box_v u \not\leq \neg v$.

**Lemma 4** *If $u \preccurlyeq \neg v =$ Left, then $\Diamond_u \Box_v u \leq v$ is impossible, and hence $u \preccurlyeq v$ cannot be Right or Both or All.*

**Proof** Similar to Lemma 3.

**Lemma 5** *If $u \preccurlyeq \neg v =$ All, then*

1. *$u \preccurlyeq v \neq$ Both and $u \preccurlyeq v \neq$ Left,*
2. *$v \preccurlyeq u \neq$ Both and $v \preccurlyeq u \neq$ Left.*

**Proof** The two parts are similar, so we only give the details of the first. If $\Box_u \Diamond_v u \leq v$ then every crisping of $u$ is inside some crisping of $v$, but since $\Box_u \Box_v u \leq \neg v$, every crisping of $u$ is outside every crisping of $v$. This can happen only if $[\![u]\!] = \varnothing$, or if $[\![u]\!] = \{\bot\}$. In either case $\Box_u \Box_v u \leq v$ holds. Thus whenever $\Box_u \Diamond_v u \leq v$, we also have $\Box_u \Box_v u \leq v$. This prevents $u \preccurlyeq v$ taking either of the values Left or Both.

**Lemma 6** *If $u \preccurlyeq \neg v =$ Both then $u \preccurlyeq v \neq$ All, and $v \preccurlyeq u \neq$ All.*

**Proof** If $u \preccurlyeq v =$ All, then $\Box_u \Box_v u \leq v$ so every crisping of $u$ is outside every crisping of $v$. But, $\Box_u \Diamond_v u \leq \neg v$ so every crisping of $u$ is outside some crisping of $v$. This can happen only if $[\![u]\!] = \varnothing$, or if $[\![u]\!] = \{\bot\}$. In either case $\Box_u \Box_v u \leq \neg v$ holds, which contradicts our assumption that $u \preccurlyeq \neg v =$ Both. A similar argument shows that $v \preccurlyeq u \neq$ All.

### 5.2 Restrictions for $\cap$-Closed Crispings

One possible semantics for egg-yolk regions is that the set of crispings is any region containing the yolk and contained within the egg. This provides an example where $[\![u]\!]$ has the property that the intersection of any two crispings is again a crisping, a situation we shall refer to as $\cap$-closed. Another example could be the crisping of the boundary of a country $C$ in the case that additional territories $t_1$ and $t_2$ are disputed between other countries $A$ and $B$ but where $C$ has an undisputed core territory $c$. In this case the set of crispings of the country $C$ might be $\{c, c \cup t_1, c \cup t_2, c \cup t_1 \cup t_2\}$. In this example the set of crispings is $\cap$-closed, but is not dense, in the sense that any region intermediate between two crispings is itself a crisping. This denseness property would be possessed by the crispings of egg-yolk regions under the semantics just mentioned.

In this subsection we examine the restrictions which are placed upon the triple $(u \preccurlyeq v, u \preccurlyeq \neg v, v \preccurlyeq u)$ by the assumption that $[\![u]\!]$ and $[\![v]\!]$ are $\cap$-closed. We shall see that these restrictions, in conjunction with ones of the previous subsection, serve to reduce the potential 216 values to 85.

**Lemma 7** *If $[\![u]\!]$ is $\cap$-closed and $u \preccurlyeq \neg v =$ All then $u \preccurlyeq v \neq$ Top and $v \preccurlyeq u \neq$ Top.*

**Proof** If $u \preccurlyeq v =$ Top then some crisping of $u$ is inside some crisping of $v$, but since $\Diamond_u \Box_v u \leq v$, some crisping of $u$ is outside every crisping of $v$. The same argument as in lemma 9 leads to a contradiction. This argument can also be used to show that $u \preccurlyeq v =$ Left is impossible, but this case has already been excluded without the assumption of $\cap$-closure in lemma 5. The demonstration that $v \preccurlyeq u \neq$ Top is similar.

**Lemma 8** *If $[\![u]\!]$ is $\cap$-closed and $u \preccurlyeq \neg v =$ Top, then neither $u \preccurlyeq v$ nor $v \preccurlyeq u$ can take any value in the set $\{$Right, Both, All$\}$.*

**Proof** If $u \preccurlyeq v$ takes one of the values Right, Both, or All, then $\Diamond_u \Box_v u \leq v$. Thus some crisping of $u$ is inside every crisping of $v$. But since $u \preccurlyeq \neg v =$ Top, we have $\Diamond_u \Diamond_v u \leq \neg v$, i.e. some crisping of $u$ is outside some crisping of $v$. So as $[\![u]\!]$ is $\cap$-closed, $\bot \in [\![u]\!]$, but then $\Diamond_u \Box_v u \leq \neg v$ which contradicts $u \preccurlyeq \neg v =$ Top.

The restriction on the possible values of $v \preccurlyeq u$ follows similarly.

**Lemma 9** *If $[\![u]\!]$ is $\cap$-closed and $u \preccurlyeq \neg v =$ Both, then neither $u \preccurlyeq v$ nor $v \preccurlyeq u$ can take a value in the set $\{$Left, Top$\}$.*

**Proof** If $\Diamond_u \Diamond_v u \leq v$ some crisping of $u$ is outside some crisping of $v$, but since $\Diamond_u \Box_v u \leq \neg v$, some crisping of $u$ is outside every crisping of $v$. Thus $u$ contains two disjoint crispings. So, as $[\![u]\!]$ is $\cap$-closed, $\bot \in [\![u]\!]$. This implies that $\Diamond_u \Box_v u \leq v$, so the cases $u \preccurlyeq v =$ Top and $u \preccurlyeq v =$ Left are impossible. Similarly, $v \preccurlyeq u \notin \{$Left, Top$\}$.

**Lemma 10** *If $[\![v]\!]$ is $\cap$-closed and $u \preccurlyeq \neg v =$ Left then $v \preccurlyeq u \notin \{$Top, Left$\}$.*

**Proof** If $v \preccurlyeq u \in \{\mathsf{Top}, \mathsf{Left}\}$ then $\Diamond_u \Diamond_v v \leq u$, but as $u \preccurlyeq \neg v = \mathsf{Left}$, we have $\Box_u \Diamond_v u \leq \neg v$. Thus some crisping of $v$ is inside some crisping of $u$ while every crisping of $u$ is outside some crisping of $v$. As $[\![v]\!]$ is $\cap$-closed, $\bot \in [\![v]\!]$ and so $v \preccurlyeq u \notin \{\mathsf{Top}, \mathsf{Left}\}$.

**Lemma 11** *If $[\![u]\!]$ is $\cap$-closed and $u \preccurlyeq \neg v =$ Right then $u \preccurlyeq v \notin \{$Top, Left$\}$.*

**Proof** Similar to lemma 10.

From these lemmas the 216 potential cases reduce to 85, as summarized in the following theorem.

**Theorem 12** *If $[\![u]\!]$ and $[\![v]\!]$ are $\cap$-closed, then the value of the triple $(u \preccurlyeq v,\ u \preccurlyeq \neg v,\ v \preccurlyeq u)$ will either be one of the 51 cases permitted by the following constraints,*

| $u \preccurlyeq v \in$ | $u \preccurlyeq \neg v =$ | $v \preccurlyeq u \in$ |
|---|---|---|
| {*All*, *None*, *Both*, *Right*} | *Right* | {*None*, *Top*, *Left*} |
| {*None*, *Top*, *Left*} | *Left* | {*All*, *None*, *Both*, *Right*} |
| {*None*, *Top*, *Right*} | *Top* | {*None*, *Top*, *Right*} |
| {*None*, *Both*, *Left*} | *Both* | {*None*, *Both*, *Left*} |
| {*None*, *All*, *Right*} | *All* | {*None*, *All*, *Right*} |

*or one of the 34 cases where $u \preccurlyeq \neg v =$ None but where either (a) $u \preccurlyeq v \neq$ All or $v \preccurlyeq u \neq$ Both or (b) $u \preccurlyeq v \neq$ Both or $v \preccurlyeq u \neq$ All.*

The theorem shows that $216 - 85 = 131$ cases can never arise. To show that all the 85 cases can actually arise is a separate task, and this is accomplished in the following section by producing explicit instances of all the cases for a particular notion of vague region.

## 6 Special Case of Egg-Yolk Regions

The classification schema using the six-valued part of relation can be applied to the specific case of egg-yolk regions. It is not necessary to assume we are working with pairs of regions which satisfy the RCC axioms, the analysis can be carried out under the assumption that we have a set of crisp regions $R$, including the empty region, which form a Boolean algebra. In this situation, the set of egg-yolk regions is given by $R^V = \{\langle r, s\rangle \in R \times R \mid r \leq s\}$. For a region $v = \langle r, s\rangle \in R^V$ the notation $\underline{v}$ will be used to denote $r$, and $\overline{v}$ to denote $s$.

Even if $R$ is based on a set of RCC regions, the elements of $R^V$ are more general than the egg-yolks studied in [LC94,CG96], since for a region, $v$, it is permitted that $\underline{v} = \overline{v}$ and that $\underline{v}$ or $\overline{v}$ may be empty. This greater generality is significant, since our analysis includes for example relations between a crisp and a non-crisp region and relations between regions where the empty crisping is permitted. Both these cases are excluded by the restrictions in [CG96].

If the semantics of egg-yolk regions is given by $[\![v]\!] = \{x \in R \mid \underline{v} \leq x \leq \overline{v}\}$, the four formulæ used to determine the value of $u \preccurlyeq v$ can all be expressed in terms of $\underline{u}$, $\underline{v}$, $\overline{v}$, and $\overline{u}$ as folows.

$$\Box_u \Box_v u \leq v \text{ iff } \overline{u} \leq \underline{v} \qquad \Diamond_u \Diamond_v u \leq v \text{ iff } \underline{u} \leq \overline{v}$$
$$\Box_u \Diamond_v u \leq v \text{ iff } \overline{u} \leq \overline{v} \qquad \Diamond_u \Box_v u \leq v \text{ iff } \underline{u} \leq \underline{v}$$

### 6.1 Realizing 85 values of $(u \preccurlyeq v, u \preccurlyeq \neg v, v \preccurlyeq u)$

The aim of this subsection is to show that all the 85 cases which are potential values of the triple $(u \preccurlyeq v, u \preccurlyeq \neg v, v \preccurlyeq u)$ do actually arise for some notion of vague region. It is not necessary to have any kind of spatial structure in the 'regions' in order to do this, simple sets are adequate. We start with the set $\{1, 2, 3, 4, 5\}$, and work with pairs $u = \langle \underline{u}, \overline{u} \rangle$ where $\underline{u} \subseteq \overline{u} \subseteq \{1, 2, 3, 4, 5\}$. The semantics is given by $[\![u]\!] = \{x \subseteq \{1, 2, 3, 4, 5\} \mid \underline{u} \subseteq x \subseteq \overline{u}\}$. The tables 3, 4, 6, 5, and table 7 detail examples of $u$ and $v$ which cover all the 85 values. It is interesting to note that all but one of these values, (Top, None, Top) could be found using only a four element set.

| $u \preccurlyeq v$ | $u \preccurlyeq \neg v$ | $v \preccurlyeq u$ | $\underline{u}$ | $\overline{u}$ | $\underline{v}$ | $\overline{v}$ |
|---|---|---|---|---|---|---|
| All | All | All | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ |
| All | All | Right | $\{\}$ | $\{\}$ | $\{\}$ | $\{1\}$ |
| All | All | None | $\{\}$ | $\{\}$ | $\{1\}$ | $\{1\}$ |
| Right | All | All | $\{\}$ | $\{1\}$ | $\{\}$ | $\{\}$ |
| None | All | All | $\{1\}$ | $\{1\}$ | $\{\}$ | $\{\}$ |
| Right | All | Right | $\{\}$ | $\{1\}$ | $\{\}$ | $\{2\}$ |
| Right | All | None | $\{\}$ | $\{1\}$ | $\{2\}$ | $\{2\}$ |
| None | All | Right | $\{1\}$ | $\{1\}$ | $\{\}$ | $\{2\}$ |
| None | All | None | $\{1\}$ | $\{1\}$ | $\{2\}$ | $\{2\}$ |

| $u \preccurlyeq v$ | $u \preccurlyeq \neg v$ | $v \preccurlyeq u$ | $\underline{u}$ | $\overline{u}$ | $\underline{v}$ | $\overline{v}$ |
|---|---|---|---|---|---|---|
| Both | Both | Both | $\{\}$ | $\{1\}$ | $\{\}$ | $\{1\}$ |
| Both | Both | Right | $\{\}$ | $\{1\}$ | $\{\}$ | $\{1, 2\}$ |
| Both | Both | None | $\{\}$ | $\{1\}$ | $\{2\}$ | $\{1, 2\}$ |
| Right | Both | Both | $\{\}$ | $\{1, 2\}$ | $\{\}$ | $\{1\}$ |
| None | Both | Both | $\{1\}$ | $\{1, 2\}$ | $\{\}$ | $\{2\}$ |
| Right | Both | Right | $\{\}$ | $\{1, 2\}$ | $\{\}$ | $\{2, 3\}$ |
| Right | Both | None | $\{\}$ | $\{1, 2\}$ | $\{3\}$ | $\{1, 3\}$ |
| None | Both | Right | $\{1\}$ | $\{1, 2\}$ | $\{\}$ | $\{2, 3\}$ |
| None | Both | None | $\{1\}$ | $\{1, 2\}$ | $\{3\}$ | $\{2, 3\}$ |

**Table 3.** Realizations of possible relations between egg-yolk pairs when $u \preccurlyeq \neg v$ is Both or All

| $u \preccurlyeq v$ | $u \preccurlyeq \neg v$ | $v \preccurlyeq u$ | $\underline{u}$ | $\overline{u}$ | $\underline{v}$ | $\overline{v}$ |
|---|---|---|---|---|---|---|
| Left | Top | Left | $\{1\}$ | $\{1,2\}$ | $\{2\}$ | $\{1,2\}$ |
| Left | Top | Top | $\{1\}$ | $\{1,2\}$ | $\{2\}$ | $\{1,2,3\}$ |
| Left | Top | None | $\{1\}$ | $\{1,2\}$ | $\{2,3\}$ | $\{1,2,3\}$ |
| Top | Top | Left | $\{1\}$ | $\{1,2,3\}$ | $\{2\}$ | $\{1,2\}$ |
| None | Top | Left | $\{1,2\}$ | $\{1,2,3\}$ | $\{3\}$ | $\{1,3\}$ |
| Top | Top | Top | $\{1\}$ | $\{1,2,3\}$ | $\{3\}$ | $\{1,3,4\}$ |
| Top | Top | None | $\{1\}$ | $\{1,2,3\}$ | $\{3,4\}$ | $\{1,3,4\}$ |
| None | Top | Top | $\{1,2\}$ | $\{1,2,3\}$ | $\{3\}$ | $\{2,3,4\}$ |
| None | Top | None | $\{1,2\}$ | $\{1,2,3\}$ | $\{3,4\}$ | $\{1,3,4\}$ |

**Table 4.** Realizations of possible relations between egg-yolk pairs when $u \preccurlyeq \neg v =$ Top

| $u \preccurlyeq v$ | $u \preccurlyeq \neg v$ | $v \preccurlyeq u$ | $\underline{u}$ | $\overline{u}$ | $\underline{v}$ | $\overline{v}$ |
|---|---|---|---|---|---|---|
| Left | Left | All | $\{1\}$ | $\{1\}$ | $\{\}$ | $\{1\}$ |
| Left | Left | Right | $\{1\}$ | $\{1\}$ | $\{\}$ | $\{1,2\}$ |
| Left | Left | None | $\{1\}$ | $\{1\}$ | $\{2\}$ | $\{1,2\}$ |
| Top | Left | All | $\{1\}$ | $\{1,2\}$ | $\{\}$ | $\{1\}$ |
| Left | Left | Both | $\{1\}$ | $\{1,2\}$ | $\{\}$ | $\{1,2\}$ |
| None | Left | All | $\{1,2\}$ | $\{1,2\}$ | $\{\}$ | $\{1\}$ |
| Top | Left | Right | $\{1\}$ | $\{1,2\}$ | $\{\}$ | $\{1,3\}$ |
| Top | Left | None | $\{1\}$ | $\{1,2\}$ | $\{3\}$ | $\{1,3\}$ |
| Top | Left | Both | $\{1\}$ | $\{1,2,3\}$ | $\{\}$ | $\{1,2\}$ |
| None | Left | Right | $\{1,2\}$ | $\{1,2\}$ | $\{\}$ | $\{2,3\}$ |
| None | Left | None | $\{1,2\}$ | $\{1,2\}$ | $\{3\}$ | $\{1,3\}$ |
| None | Left | Both | $\{1,2\}$ | $\{1,2,3\}$ | $\{\}$ | $\{2,3\}$ |

**Table 5.** Realizations of possible relations between egg-yolk pairs when $u \preccurlyeq \neg v =$ Left

| $u \preccurlyeq v$ | $u \preccurlyeq \neg v$ | $v \preccurlyeq u$ | $\underline{u}$ | $\overline{u}$ | $\underline{v}$ | $\overline{v}$ |
|---|---|---|---|---|---|---|
| All | Right | Left | $\{\}$ | $\{1\}$ | $\{1\}$ | $\{1\}$ |
| All | Right | Top | $\{\}$ | $\{1\}$ | $\{1\}$ | $\{1,2\}$ |
| All | Right | None | $\{\}$ | $\{1\}$ | $\{1,2\}$ | $\{1,2\}$ |
| Right | Right | Left | $\{\}$ | $\{1,2\}$ | $\{1\}$ | $\{1\}$ |
| Both | Right | Left | $\{\}$ | $\{1,2\}$ | $\{1\}$ | $\{1,2\}$ |
| None | Right | Left | $\{1\}$ | $\{1,2\}$ | $\{2\}$ | $\{2\}$ |
| Both | Right | Top | $\{\}$ | $\{1,2\}$ | $\{1\}$ | $\{1,2,3\}$ |
| Right | Right | Top | $\{\}$ | $\{1,2\}$ | $\{2\}$ | $\{2,3\}$ |
| Right | Right | None | $\{\}$ | $\{1,2\}$ | $\{2,3\}$ | $\{2,3\}$ |
| Both | Right | None | $\{\}$ | $\{1,2\}$ | $\{2,3\}$ | $\{1,2,3\}$ |
| None | Right | Top | $\{1\}$ | $\{1,2\}$ | $\{2\}$ | $\{2,3\}$ |
| None | Right | None | $\{1\}$ | $\{1,2\}$ | $\{2,3\}$ | $\{2,3\}$ |

**Table 6.** Realizations of possible relations between egg-yolk pairs when $u \preccurlyeq \neg v =$ Right

| $u \preccurlyeq v$ | $u \preccurlyeq \neg v$ | $v \preccurlyeq u$ | $\underline{u}$ | $\overline{u}$ | $\underline{v}$ | $\overline{v}$ |
|---|---|---|---|---|---|---|
| All | None | All | $\{1\}$ | $\{1\}$ | $\{1\}$ | $\{1\}$ |
| All | None | Right | $\{1\}$ | $\{1\}$ | $\{1\}$ | $\{1,2\}$ |
| All | None | None | $\{1\}$ | $\{1\}$ | $\{1,2\}$ | $\{1,2\}$ |
| Right | None | All | $\{1\}$ | $\{1,2\}$ | $\{1\}$ | $\{1\}$ |
| Both | None | Both | $\{1\}$ | $\{1,2\}$ | $\{1\}$ | $\{1,2\}$ |
| All | None | Left | $\{1\}$ | $\{1,2\}$ | $\{1,2\}$ | $\{1,2\}$ |
| None | None | All | $\{1,2\}$ | $\{1,2\}$ | $\{1\}$ | $\{1\}$ |
| Left | None | All | $\{1,2\}$ | $\{1,2\}$ | $\{1\}$ | $\{1,2\}$ |
| Both | None | Right | $\{1\}$ | $\{1,2\}$ | $\{1\}$ | $\{1,2,3\}$ |
| All | None | Top | $\{1\}$ | $\{1,2\}$ | $\{1,2\}$ | $\{1,2,3\}$ |
| Right | None | Right | $\{1\}$ | $\{1,2\}$ | $\{1\}$ | $\{1,3\}$ |
| Right | None | None | $\{1\}$ | $\{1,2\}$ | $\{1,3\}$ | $\{1,3\}$ |
| Both | None | None | $\{1\}$ | $\{1,2\}$ | $\{1,3\}$ | $\{1,2,3\}$ |
| Right | None | Both | $\{1\}$ | $\{1,2,3\}$ | $\{1\}$ | $\{1,2\}$ |
| Right | None | Left | $\{1\}$ | $\{1,2,3\}$ | $\{1,2\}$ | $\{1,2\}$ |
| Both | None | Left | $\{1\}$ | $\{1,2,3\}$ | $\{1,2\}$ | $\{1,2,3\}$ |
| Left | None | Right | $\{1,2\}$ | $\{1,2\}$ | $\{1\}$ | $\{1,2,3\}$ |
| None | None | Right | $\{1,2\}$ | $\{1,2\}$ | $\{2\}$ | $\{2,3\}$ |
| None | None | None | $\{1,2\}$ | $\{1,2\}$ | $\{2,3\}$ | $\{2,3\}$ |
| Left | None | None | $\{1,2\}$ | $\{1,2\}$ | $\{2,3\}$ | $\{1,2,3\}$ |
| Top | None | All | $\{1,2\}$ | $\{1,2,3\}$ | $\{1\}$ | $\{1,2\}$ |
| Left | None | Both | $\{1,2\}$ | $\{1,2,3\}$ | $\{1\}$ | $\{1,2,3\}$ |
| None | None | Both | $\{1,2\}$ | $\{1,2,3\}$ | $\{2\}$ | $\{2,3\}$ |
| None | None | Left | $\{1,2\}$ | $\{1,2,3\}$ | $\{2,3\}$ | $\{2,3\}$ |
| Left | None | Left | $\{1,2\}$ | $\{1,2,3\}$ | $\{2,3\}$ | $\{1,2,3\}$ |
| Both | None | Top | $\{1\}$ | $\{1,2,3\}$ | $\{1,2\}$ | $\{1,2,3,4\}$ |
| Right | None | Top | $\{1\}$ | $\{1,2,3\}$ | $\{1,3\}$ | $\{1,3,4\}$ |
| Left | None | Top | $\{1,2\}$ | $\{1,2,3\}$ | $\{2,3\}$ | $\{1,2,3,4\}$ |
| None | None | Top | $\{1,2\}$ | $\{1,2,3\}$ | $\{2,3\}$ | $\{2,3,4\}$ |
| Top | None | Right | $\{1,2\}$ | $\{1,2,3\}$ | $\{1\}$ | $\{1,2,4\}$ |
| Top | None | None | $\{1,2\}$ | $\{1,2,3\}$ | $\{2,4\}$ | $\{1,2,4\}$ |
| Top | None | Both | $\{1,2\}$ | $\{1,2,3,4\}$ | $\{1\}$ | $\{1,2,3\}$ |
| Top | None | Left | $\{1,2\}$ | $\{1,2,3,4\}$ | $\{2,3\}$ | $\{1,2,3\}$ |
| Top | None | Top | $\{1,2\}$ | $\{1,2,3,4\}$ | $\{2,4\}$ | $\{1,2,4,5\}$ |

**Table 7.** Realizations of possible relations between egg-yolk pairs when $u \preccurlyeq \neg v =$ None

### 6.2 Relationship to the Cohn and Gotts Egg-Yolk Classification

Cohn and Gotts [CG96] document 46 possible relations between egg-yolks pairs. Their classification is based on the relative positions of the eggs and yolks and, unlike the present paper, does not derive the classification from considerations of the semantics of vague regions. However the 85 possible values of $(u \preccurlyeq v, u \preccurlyeq \neg v, v \preccurlyeq \neg u)$ identified in our analysis do include exactly the 46 cases identified by Cohn and Gotts. Our 85 cases reduce to 46 if it is assumed that empty crispings are not permitted, and that the yolk is a proper subset of the egg.

In [CG96] the relationship of $u$ to $v$ is determined by the quadruple of RCC5 relationships between the pairs $(\underline{u}, \underline{v})$, $(\underline{u}, \overline{v})$, $(\overline{u}, \underline{v})$, and $(\overline{u}, \overline{v})$. We have shown earlier that RCC5 can be derived from concepts of part and complement alone. If this approach is used we see that the Cohn and Gotts classification arises from four triples of values, that is each triple to determine the RCC5 classification for one element of their quadruple. The resulting twelve boolean values can be arranged in a table, and this establishes the precise correspondence between our six-valued approach and that of [CG96].

| | | | |
|---|---|---|---|
| $\overline{u} \leq \underline{v}$ | $\overline{u} \leq \neg\underline{v}$ | $\underline{v} \leq \overline{u}$ | RCC5$^+(\overline{u}, \underline{v})$ |
| $\overline{u} \leq \overline{v}$ | $\overline{u} \leq \neg\overline{v}$ | $\overline{v} \leq \overline{u}$ | RCC5$^+(\overline{u}, \overline{v})$ |
| $\underline{u} \leq \underline{v}$ | $\underline{u} \leq \neg\underline{v}$ | $\underline{v} \leq \underline{u}$ | RCC5$^+(\underline{u}, \underline{v})$ |
| $\underline{u} \leq \overline{v}$ | $\underline{u} \leq \neg\overline{v}$ | $\overline{v} \leq \underline{u}$ | RCC5$^+(\underline{u}, \overline{v})$ |

$(u \preccurlyeq v,\ u \preccurlyeq \neg v,\ v \preccurlyeq u)$

The four values in each column determine one of the three components of our approach to relations between vague regions. However the same twelve values, read as rows provide exactly the four components needed to determine the relationship between two vague regions in Cohn and Gotts approach.

## 7 Conclusions and Further Work

This paper has introduced a new and versatile technique for producing systems of spatial relations. The technique is based only on the fundamental concepts of part and of complement, and can be used in constructing formal ontologies for spatial regions in qualitative spatial reasoning and in geographical information systems.

The viability of the technique has been demonstrated by showing how it leads to

1. a classification extending RCC5 which allows regions to be empty,
2. a similar extension to RCC8,
3. a classification for vague regions, which includes as a special case, the classification of relationships between egg-yolk regions.

The derivation of the classification for vague regions has led to the identification of a six-valued part of relation as being appropriate for vague regions. This six-valued relation appears to be of some independent interest, and contributes to our understanding of the mereology of vague regions.

There are several directions for further work based on the ideas in this paper. A particularly promising area for further work would be to consider relations between abstract graphs in the context of work on discrete representations of space [Ste00b]. A data model for graphs in spatial databases has been investigated by Erwig and Güting [EG94], and a discussion of notions of part and complement for graphs appears in work by Stell and Worboys [SW97]. As there is more than one notion of complement for graphs, some work is needed to investigate which of these is most appropriate to produce a scheme for relations between graphs. The ideas on vague regions in this paper could then be applied to handle vague graphs using one of the notions of vague graphs suggested in [Ste99].

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